AD-A033 758 UTAH UNIV SALT LAKE CITY DEPT OF MATHEMATICS F/G 20/1
SPECTRAL AND ASYMPTOTIC ANALYSIS OF ACOUSTIC WAVE PROPAGATION.(U)
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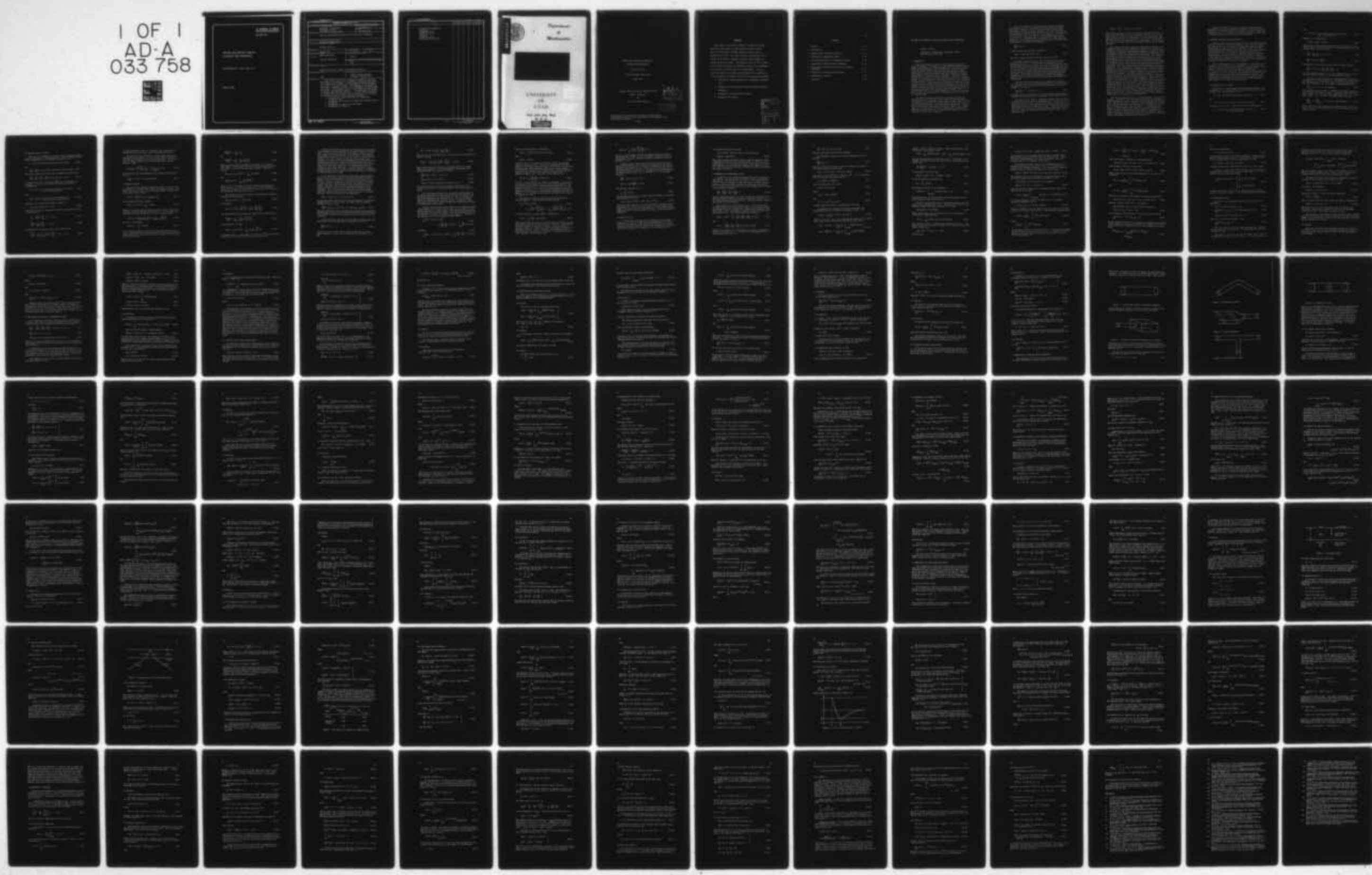

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AD-A033 758

SPECTRAL AND ASYMPTOTIC ANALYSIS OF ACOUSTIC WAVE PROPAGATION

UTAH UNIVERSITY, SALT LAKE CITY

AUGUST 1976

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Department of Mathematics University of Utah Salt Lake City, Utah 84112	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP Not applicable

3. REPORT TITLE

Spectral and Asymptotic Analysis of Acoustic Wave Propagation

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Summary Report

5. AUTHOR(S) (First name, middle initial, last name)

Wilcox, Calvin H.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
August 1976	iii + 89	44

8a. CONTRACT OR GRANT NO. N00014-76-C-0276 b. PROJECT NO. Task No. NR-041-370 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) TR #29 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) none

10. DISTRIBUTION STATEMENT

Approved for public release, distribution unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
none	Office of Naval Research Arlington, Virginia 22217

13. ABSTRACT

This report is the written version of a series of lectures presented by the author at a NATO advanced Studies Institute held at the University of Liège, Belgium, during the period September 6-17, 1976. The report presents an exposition of the results of the author's research on acoustic wave propagation during the last five years. The principal result of this research is the "method of spectral and asymptotic analysis" for determining the structure of acoustic waves in unbounded media. In this report the method is explained and illustrated by its application to the following four classes of acoustic wave propagation problems.

1. Scattering by bounded obstacles in a homogeneous unlimited fluid.
2. Propagation and scattering in simple and compound tubular waveguides.
3. Propagation in plane stratified fluids.
4. Propagation in crystals.

Unclassified
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Acoustic wave propagation scattering homogeneous fluid stratified fluid waveguide anisotropic solid asymptotic wave function						

Unclassified
Security Classification

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Technical Summary Report #29

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Prepared under Contract No. N00014-76-C-0276

Task No. NR-041-370

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Office of Naval Research

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CONTENTS

SPECTRAL AND ASYMPTOTIC ANALYSIS OF ACOUSTIC WAVE PROPAGATION

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1. INTRODUCTION

Classical theories of acoustic wave propagation provide a wealth of examples of boundary value problems for evolution partial differential equations. These problems may be described categorically as initial-boundary value problems for certain systems of linear hyperbolic partial differential equations with variable coefficients. However, the known existence, uniqueness and regularity theorems for these problems are only a first step toward understanding the structure of the solutions. To obtain a deeper insight it is essential to discover how the nature of the solutions changes with the geometry of the boundary and with the coefficients. An examination of recent scientific literature on acoustics reveals a great variety of physically distinct phenomena. Examples include phenomena associated with acoustic wave propagation in stratified fluids, anisotropic solids such as crystals and man-made composites, open and closed waveguides, periodic media and many others. A theory which treats all of these phenomena on the same footing can provide only the most superficial information about the structure of acoustic waves.

The purpose of these lectures is to present a method for determining the structure of acoustic waves in unbounded media. The method will be explained in the context of four specific classes of propagation problems. No attempt will be made to formulate the most general problem that can be analyzed by the method. Indeed, such a formulation would necessarily be too abstract to be useful. However, it will be clear from the examples that the method is applicable to many other wave propagation problems, both in acoustics and in other areas of physics.

It will be helpful to outline the main steps of the method here before passing to a detailed discussion of specific cases. The method is based on the fact that the states of an acoustic medium which occupies a spatial domain $\Omega \subset R^3$ can be described by the elements of a Hilbert space $\mathcal{H}$ of functions on Ω. The evolution of an acoustic wave in the medium is then described by a curve $t \to u(t,\cdot) \in \mathcal{H}$. Moreover, there is a selfadjoint real positive operator A on $\mathcal{H}$, determined by the geometry of Ω and the physical properties of the medium, such that the evolution of acoustic waves in the medium is governed by the equation

$$\frac{d^2u}{dt^2} + Au = 0 \tag{1.1}$$

It follows that the evolution is given by

$$u(t,\cdot) = \mathrm{Re}\,\{\exp\,(-itA^{1/2})h\} \tag{1.2}$$

where $h \in \mathcal{H}$ characterizes the initial state of the wave.

The spectral theorem may be used to construct the solution operator $\exp\,(-itA^{1/2})$. However, the very generality of this theorem implies that it can give little specific information about the structure of the wave functions $u(t,x)$. Accordingly, the next step in the method is to construct an eigenfunction expansion for A. In each of the cases discussed below A has a purely continuous spectrum and the eigenfunctions are therefore generalized eigenfunctions. They define a complete set of steady-state modes of propagation of the medium and the most general time-dependent acoustic wave in $\mathcal{H}$ can be constructed as a spectral integral over these modes.

The final step in the method is an asymptotic analysis for $t \to \infty$ of the spectral integral representing $u(t,x)$. The result is an asymptotic wave function $u^\infty(t,x)$ which approximates $u(t,x)$ in $\mathcal{H}$ when $t \to \infty$; that is,

$$\lim_{t\to\infty} \| u(t,\cdot) - u^\infty(t,\cdot)\|_{\mathcal{H}} = 0 \tag{1.3}$$

Stronger forms of convergence can also be proved under appropriate supplementary hypotheses about the medium and its initial state.

The result (1.3) offers a fundamental insight into the nature of transient acoustic waves in unbounded media. For it is found in each case that the form of the asymptotic wave function $u^\infty(t,x)$ is determined entirely by the geometry of the domain Ω and the physical characteristics of the medium that fills it. Only the fine structure of $u^\infty(t,x)$ depends on the initial state of the wave. Thus in the simple case of a homogeneous fluid filling R^3, $u^\infty(t,x)$ is a spherical wave:

$$u^{\infty}(t,x) = F(r - t,\theta)/r, \ x = r\theta, \ |\theta| = 1 \tag{1.4}$$

The initial state affects only the shape of the profile $F(\tau,\theta)$. In other cases the form of $u^{\infty}(t,x)$ is entirely different, but in each case the form of $u^{\infty}(t,x)$ is determined solely by the geometry and physical characteristics of the medium. In each case $u^{\infty}(t,x)$ gives the final form of any transient wave in the medium. The details of how the wave is excited have only a secondary effect on the ultimate waveform.

The remainder of these lectures is organized as follows. The fundamental boundary value problems of acoustics are formulated in section 2. The spectral and asymptotic analysis of the four classes of propagation problems is presented in sections 3 through 8. The four classes, which are physically quite different, were chosen to illustrate the flexibility and scope of the method. In each of the four classes there is a special case for which, because of additional symmetry, the eigenfunctions can be constructed explicitly. The remaining cases of the class are then treated as perturbations of the special case. When used in this context, perturbation theory is usually called the steady-state, or time-dependent, theory of scattering. The first class of problems treated below corresponds physically to the scattering of acoustic waves by bounded obstacles immersed in a homogeneous fluid. Mathematically, it is an initial-boundary problem for the d'Alembert equation in an exterior domain $\Omega \subset R^3$ ($R^3-\Omega$ compact). The simple special case where $\Omega = R^3$ is treated in section 3 and the general case in section 4. The second class of problems deals with tubular waveguides. Thus Ω is the union of a bounded domain and a finite number of semi-infinite cylinders. The special case of a single cylinder is treated in section 5 and the general case in section 6. The third class of problems, treated in section 7, deals with acoustic wave propagation in plane stratified fluids filling a half-space. Here the novel feature is the possibility of the trapping of waves by total internal reflection. The fourth and final class of problems, dealing with acoustic waves in crystalline solids, is discussed in section 8. The new feature in this case is the anisotropy which has a profound effect on the form of the asymptotic wave functions.

The results presented below are based primarily on the author's research. Sections 3 and 4 are based on the author's monograph on "Scattering Theory for the d'Alembert Equation in Exterior Domains" [42]. The spectral theory of acoustic wave propagation and scattering in tubular waveguides was developed by C. Goldstein [9-12] and by W. C. Lyford [21,22]. More recently, J. C. Guillot and the author [13] have developed the theory for domains Ω which are the union of a bounded domain and a finite number of cylinders and cones. Sections 5 and 6 present spectral and scattering theory for tubular waveguides following the plan of [13]. Sections 7 and 8 are based on the author's publications [39,40,43,44].

The goal of these lectures is to provide an introduction to the method of spectral and asymptotic analysis of wave propagation. Therefore, the lectures emphasize concepts and results, rather than techniques of proof. Proofs of the results given here may be found in the references listed at the end of the lectures.

2. BOUNDARY VALUE PROBLEMS OF ACOUSTICS

Acoustic waves are the mechanical vibrations of small amplitude that are observed in all forms of matter. The classical equations of acoustics are the linear partial differential equations which govern small perturbations of the equilibrium states of matter. Derivations of these equations from the laws of mechanics, together with a discussion of their range of validity, may be found in [3,4,8,20,31]. In this section the equations and their physical interpretation are reviewed briefly and the principal boundary value problems for them are formulated and discussed. Applications of the equations to particular classes of acoustic wave propagation problems are developed in sections 3 through 8.

The following notation is used throughout the remainder of the lectures. $t \in R$ denotes a time coordinate. $x = (x_1, x_2, x_3) \in R^3$ denote Cartesian coordinates of a point in Euclidean space. $\Omega \subset R^3$ denotes a domain in R^3 and $\partial\Omega$ denotes the boundary of Ω. $\nu = (\nu_1, \nu_2, \nu_3) = \nu(x)$ denotes the unit exterior normal vector to $\partial\Omega$ at points $x \in \partial\Omega$ where it exists. The equations of acoustics are written below in the notation of Cartesian tensor analysis. In particular, the summation convention is used. Acoustic waves in fluids (gases and liquids) and solids are discussed separately. The simpler case of fluids is treated first.

2.1 Acoustic waves in fluids

The case of an inhomogeneous fluid occupying a domain $\Omega \subset R^3$ is considered. The propagation of acoustic waves in such a fluid is governed by two functions of $x \in \Omega$:

$\rho = \rho(x)$, the equilibrium density of the fluid (2.1)

and

$c = c(x)$, the local speed of sound in the fluid (2.2)

The state of the acoustic field in the fluid is determined by

$v_j = v_j(t,x)$, the velocity field of the fluid at time t and position x (2.3) and

$$p = p(t,x), \text{ the pressure field of the fluid at time } t \text{ and position } x \tag{2.4}$$

Moreover, it is assumed that

$$p(t,x) = p_0(x) + u(t,x) \tag{2.5}$$

where $p_0(x)$ is the equilibrium pressure of the fluid and $u(t,x)$ remains small. With this notation the equations satisfied by the acoustic field in the fluid are

$$\frac{\partial v_j}{\partial t} + \frac{1}{\rho(x)} \frac{\partial u}{\partial x_j} = 0, \; j = 1,2,3 \tag{2.6}$$

$$\frac{\partial u}{\partial t} + c^2(x)\, \rho(x) \frac{\partial v_j}{\partial x_j} = 0 \tag{2.7}$$

Elimination of the velocity field gives the single equation

$$\frac{\partial^2 u}{\partial t^2} - c^2(x)\, \rho(x) \frac{\partial}{\partial x_j} \left(\frac{1}{\rho(x)} \frac{\partial u}{\partial x_j} \right) = 0 \tag{2.8}$$

for the pressure increment $u = p - p_0$. Moreover, if u is known then the velocity field v_j can be calculated from (2.6).

The wave equation (2.8) must be supplemented by a boundary condition at the fluid boundary $\partial\Omega$. Two physically distinct cases are considered here. The first case is that of a free boundary $\partial\Omega$. Here the pressure at the boundary is unperturbed; that is,

$$u\Big|_{\partial\Omega} = 0 \text{ if } \partial\Omega \text{ is a free boundary} \tag{2.9}$$

This condition is often used to represent an air-water interface in the theory of underwater sound. The second case is that of a rigid boundary $\partial\Omega$. Here the normal component of the fluid velocity must vanish: $\nu_j v_j = 0$ on $\partial\Omega$. It follows from (2.6) that

$$\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \nu_j \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega} = 0 \text{ if } \partial\Omega \text{ is a rigid boundary} \tag{2.10}$$

The solvability of the boundary value problems (2.8), (2.9) and (2.8), (2.10) is discussed below, after the discussion of acoustic waves in solids.

2.2 Acoustic waves in solids

The case of an inhomogeneous elastic solid occupying a domain $\Omega \subset R^3$ is considered. The propagation of acoustic waves in such a solid is governed by the following functions of $x \in \Omega$:

$$\rho = \rho(x), \text{ the equilibrium density of the solid} \tag{2.11}$$

and

$$c^{jk}_{\ell m} = c^{jk}_{\ell m}(x), \text{ the stress-strain tensor for the solid} \tag{2.12}$$

The stress-strain tensor must have the symmetry properties [4].

$$c^{jk}_{\ell m} = c^{kj}_{\ell m} = c^{kj}_{m\ell} = c^{m\ell}_{kj} \text{ for all } j,k,\ell,m = 1,2,3 \tag{2.13}$$

It follows that the 81 components $c^{jk}_{\ell m}(x)$ are determined by 21 functions. The state of the acoustic field in the solid is determined by

$$u_j = u_j(t,x), \text{ the displacement field of the solid at time } t \text{ and position } x \tag{2.14}$$

and

$$\sigma_{jk} = \sigma_{jk}(t,x), \text{ the stress tensor field of the solid at time } t \text{ and position } x \tag{2.15}$$

Moreover, the stress tensor field is symmetric:

$$\sigma_{jk} = \sigma_{kj} \text{ for all } j,k = 1,2,3 \tag{2.16}$$

With this notation the equations satisfied by the acoustic field in the solid are

$$\sigma_{jk} = c^{\ell m}_{jk}(x) \frac{\partial u_\ell}{\partial x_m}, \; j,k = 1,2,3 \tag{2.17}$$

$$\frac{\partial^2 u_j}{\partial t^2} = \frac{1}{\rho(x)} \frac{\partial \sigma_{jk}}{\partial x_k}, \; j = 1,2,3 \tag{2.18}$$

Elimination of the stress tensor gives the equations

$$\frac{\partial^2 u_j}{\partial t^2} - \frac{1}{\rho(x)} \frac{\partial}{\partial x_k} \left(c^{\ell m}_{jk}(x) \frac{\partial u_\ell}{\partial x_m} \right) = 0, \; j = 1,2,3 \tag{2.19}$$

for the displacement field u_j. Moreover, if u_j is known then the stress tensor field σ_{jk} can be calculated from (2.17).

The wave equation (2.19) must be supplemented by boundary conditions at the boundary $\partial\Omega$ of the solid. Only the cases of free and rigid boundaries will be considered here. In the first case the normal component of the stress must vanish at the boundary. Hence

$$\sigma_{jk}\nu_k\Big|_{\partial\Omega} = c_{jk}^{\ell m}\frac{\partial u_\ell}{\partial x_m}\nu_k\Big|_{\partial\Omega} = 0 \text{ if } \partial\Omega \text{ is a free boundary} \tag{2.20}$$

In the second case the displacement must vanish at the boundary; that is,

$$u_j\Big|_{\partial\Omega} = 0 \text{ if } \partial\Omega \text{ is a rigid boundary} \tag{2.21}$$

2.3 Energy integrals

One of the most important formal properties of the equations of acoustics is the existence of quadratic energy integrals. The first order system (2.6), (2.7) for acoustic waves in fluids has the quadratic energy density

$$\eta(t,x) = \frac{1}{2}\left\langle\rho(x)v_jv_j + \frac{1}{c^2(x)\rho(x)}u^2\right\rangle \tag{2.22}$$

and corresponding energy integral

$$E(v_1,v_2,v_3,u,K,t) = \int_K \eta(t,x)dx \tag{2.23}$$

where $dx = dx_1dx_2dx_3$ denotes Lebesgue measure in R^3. The energy density for the derived field $v_j' = \partial v_j/\partial t$, $u' = \partial u/\partial t$, which also satisfies the field equations (2.6), (2.7), can be written

$$\eta'(t,x) = \frac{1}{2}\left\langle\frac{1}{\rho(x)}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j} + \frac{1}{c^2(x)\rho(x)}\left(\frac{\partial u}{\partial t}\right)^2\right\rangle \tag{2.24}$$

by (2.6). The integral

$$E(u,K,t) = \int_K \eta'(t,x)dx \tag{2.25}$$

is an energy integral for solutions of the scalar wave equation (2.8). The importance of these integrals in the theory of acoustic waves derives from the conservation laws for them. In differential form they state that

$$\frac{\partial \eta(t,x)}{\partial t} = -\frac{\partial}{\partial x_j}(uv_j) \tag{2.26}$$

and

$$\frac{\partial \eta'(t,x)}{\partial t} = \frac{\partial}{\partial x_j}\left(\frac{1}{\rho(x)}\frac{\partial u}{\partial t}\frac{\partial u}{\partial x_j}\right) \tag{2.27}$$

These equations follow immediately from (2.6), (2.7) and the definitions. The integral forms of the conservation laws follow from (2.26), (2.27) and the divergence theorem. They may be written

$$dE(v_1,v_2,v_3,u,K,t)/dt = -\int_{\partial K} u(\nu_j v_j)\,dS \tag{2.28}$$

and

$$dE(u,K,t)/dt = \int_{\partial K} \frac{1}{\rho(x)}\frac{\partial u}{\partial t}\frac{\partial u}{\partial \nu}\,dS \tag{2.29}$$

where $K \subset R^3$ is any domain for which the divergence theorem is valid and dS is the element of area on ∂K. In particular, if $u(t,x)$ is a solution of (2.8) which satisfies (2.9) or (2.10) then (2.29) implies that $dE(u,\Omega,t)/dt = 0$.

The equations for acoustic waves in solids have an analogous quadratic energy integral

$$E(u_1,u_2,u_3,K,t) = \int_K \eta(t,x)\,dx \tag{2.30}$$

with density

$$\eta(t,x) = \frac{1}{2}\left\langle \rho(x)\frac{\partial u_j}{\partial t}\frac{\partial u_j}{\partial t} + c_{jk}^{\ell m}(x)\frac{\partial u_j}{\partial x_k}\frac{\partial u_\ell}{\partial x_m}\right\rangle \tag{2.31}$$

The corresponding conservation law, which follows from (2.19), is

$$\frac{\partial \eta(t,x)}{\partial t} = \frac{\partial}{\partial x_k}\left(c_{jk}^{\ell m}(x)\frac{\partial u_\ell}{\partial x_m}\frac{\partial u_j}{\partial t}\right) \tag{2.32}$$

in differential form and

$$dE(u_1,u_2,u_3,K,t)/dt = \int_{\partial K} (\sigma_{jk}\nu_k)\frac{\partial u_j}{\partial t}\,dS \tag{2.33}$$

in integral form. In particular, solutions of (2.19) which satisfy (2.20) or (2.21) also satisfy $dE(u_1,u_2,u_3,\Omega,t)/dt = 0$.

The preceding remarks emphasize the mathematical relationship of the quadratic energy integrals to the field equations of acoustics. The term "energy" has been used because in certain cases the integrals can be interpreted as the portion of the energy of the acoustic field that is in the set K at time t. This interpretation is not always correct because the linear equations of acoustics are only a first-order approximation to more complicated nonlinear equations and the energy densities defined above are second-order quantities. Hence, it is possible that other second-order terms which were dropped in the linearization should be included in the energy densities. A correct calculation of the energy must begin with the original nonlinear problem. A discussion of these problems may be found in [8,31] for the case of fluids and in [4] for the case of solids.

It is important to realize that the energy integrals defined above play an essential role in the theory of acoustic fields, whether or not they represent the actual physical energy of the fields. Indeed, it was shown in [33] and [34] that the existence of these integrals implies the existence and uniqueness of solutions to the basic initial-boundary value problems for acoustic fields. Moreover, recent work on eigenfunction expansions and scattering theory makes use of Hilbert spaces based on energy integrals. The one indispensible hypothesis that must be made is that the quadratic forms (2.22) or (2.24) and (2.31) be positive definite. For (2.22) and (2.24) this means that

$$\rho(x) > 0 \text{ and } c^2(x) > 0 \text{ for all } x \in \Omega \tag{2.34}$$

In any case, these hypotheses are essential because of the physical interpretation of $\rho(x)$ and $c(x)$. The form (2.31) is positive definite if $\rho(x) > 0$ and

$$c_{jk}^{\ell m}(x)\, \xi_{\ell m}\, \xi_{jk} > 0 \text{ for all } x \in \Omega \text{ and } \xi_{\ell m} = \xi_{m\ell} \not\equiv 0 \tag{2.35}$$

The last condition can also be expressed by means of the well-known determinantal criteria for a quadratic form to be positive definite. It is assumed throughout these lectures that (2.34) and (2.35) are satisfied.

It has been shown that the acoustic fields in both fluids and solids satisfy partial differential equations of the form

$$\frac{\partial^2 u}{\partial t^2} + Au = 0 \tag{2.36}$$

where A is a second order partial differential operator in the space variables. In the case of fluids $u(t,x) \in R$, $Au(t,x) \in R$ and

$$Au = -c^2(x)\ \rho(x)\ \frac{\partial}{\partial x_j}\left(\frac{1}{\rho(x)}\ \frac{\partial u}{\partial x_j}\right) \tag{2.37}$$

while in the case of solids $u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x)) \in R^3$, $Au(t,x) \in R^3$ and

$$(Au)_j = -\frac{1}{\rho(x)}\ \frac{\partial}{\partial x_k}\left(c_{jk}^{\ell m}(x)\ \frac{\partial u_\ell}{\partial x_m}\right),\ j = 1,2,3 \tag{2.38}$$

Thus in both cases the evolution of acoustic waves in a medium which fills a domain $\Omega \subset R^3$ is described by the solution of an initial-boundary value problem of the form

$$\frac{\partial^2 u}{\partial t^2} + Au = 0 \text{ for } t > 0,\ x \in \Omega \tag{2.39}$$

$$Bu = 0 \text{ for } t \geq 0,\ x \in \partial\Omega \tag{2.40}$$

$$u(0,x) = f(x) \text{ and } \partial u(0,x)/\partial t = g(x) \text{ for } x \in \Omega \tag{2.41}$$

Here (2.40) represents one of the boundary conditions (2.9), (2.10) in the case of a fluid and (2.20), (2.21) in the case of a solid.

It is interesting to note that the positive definiteness of the energy densities, hypothesized above on physical grounds, implies the hyporbolicity of the equation (2.36). It follows that the initial-boundary value problem (2.39) - (2.41) has compact domains of dependence and influence [6,33]. In physical terms this means that acoustic waves propagate into undisturbed portions of a medium with finite speed.

A simple and rigorous solution theory for the initial-boundary value problem (2.39) - (2.41) can be based on the theory of self-adjoint operators in Hilbert space. This possibility follows from the divergence theorem which implies the formal selfadjointness of the operators A relative to suitable inner products. Indeed, for the operator (2.37) the divergence theorem implies

$$\int_\Omega \overline{Au}\ v\ c^{-2}(x)\ \rho^{-1}(x)dx$$
$$= \int_\Omega \frac{\overline{\partial u}}{\partial x_j}\ \frac{\partial v}{\partial x_j}\ \rho^{-1}(x)dx - \int_{\partial\Omega} \frac{\overline{\partial u}}{\partial \nu}\ v\ \rho^{-1}(x)dS \tag{2.42}$$

and hence (2.43)

$$\int_\Omega \{\overline{Au}\ v - \overline{u}\ Av\}c^{-2}(x)\ \rho^{-1}(x)dx = \int_{\partial\Omega} \{\overline{u}\ \frac{\partial v}{\partial \nu} - \frac{\overline{\partial u}}{\partial \nu}\ v\}\ \rho^{-1}(x)dS$$

Thus if an inner product is defined by

$$(u,v) = \int_\Omega \overline{u(x)}\; v(x)\; c^{-2}(x)\; \rho^{-1}(x)dx \tag{2.44}$$

then

$$(Au,v) = (u,Av) \tag{2.45}$$

for all u and v in the domain of A which satisfy the boundary condition (2.9) or (2.10). Moreover, (2.44) defines the inner product in the Hilbert space $\mathcal{H} = L_2(\Omega, c^{-2}(x)\rho^{-1}(x)dx)$ of functions on Ω which are square-integrable with respect to the measure $c^{-2}(x)\rho^{-1}(x)dx$. Hence (2.45) implies that A, acting in the classical sense on functions which satisfy (2.9) or (2.10), is a symmetric operator in $\mathcal{H}$. Moreover, (2.42) implies that

$$(Au,u) = \int_\Omega \frac{\overline{\partial u}}{\partial x_j} \frac{\partial u}{\partial x_j}\, \rho^{-1}(x)dx \geq 0 \tag{2.46}$$

for all u in the domain of A. Hence A is positive. It was shown in [42] and [43] how the domain of A could be enlarged to obtain an extension A of A which is selfadjoint and positive in $\mathcal{H}$. The boundary condition (2.9) or (2.10) is incorporated into the definition of the domain of A. Moreover, the construction provides a meaningful generalization of the boundary conditions for arbitrary domains $\Omega \subset R^3$. The precise definitions and results are reviewed in sections 3-7 below.

The operator (2.38) for acoustic waves in solids can be treated similarly. The divergence theorem implies

$$\int_\Omega \overline{(Au)_j}\; v_j\; \rho(x)dx \tag{2.47}$$
$$= \int_\Omega c^{\ell m}_{jk}(x)\; \frac{\overline{\partial u_\ell}}{\partial x_m} \frac{\partial v_j}{\partial x_k}\, dx - \int_{\partial\Omega} \overline{\left(c^{\ell m}_{jk}(x)\; \frac{\partial u_\ell}{\partial x_m}\; \nu_k\right)}\; v_j dS$$

It follows that if an inner product is defined by

$$(u,v) = \int_\Omega \overline{u_j(x)}\; v_j(x)\; \rho(x)dx \tag{2.48}$$

then (2.45) holds for all u and v in the domain of A which satisfy the boundary condition (2.20) or (2.21). Moreover, (2.48) defines the inner product in the Hilbert space $\mathcal{H} = L_2(\Omega, C^3, \rho(x)dx)$ of functions from Ω to C^3 which are square integrable with respect to the measure $\rho(x)dx$. Hence A, acting in the classical sense on functions which satisfy (2.20) or (2.21), is a symmetric operator in $\mathcal{H}$. Moreover, (2.47) implies that

$$(Au,u) = \int_\Omega c_{jk}^{\ell m}(x) \overline{\frac{\partial u_\ell}{\partial x_m}} \frac{\partial u_j}{\partial x_k} dx \geq 0 \tag{2.49}$$

for all u in the domain of $\mathring{A}$ by the assumed positivity of the energy density, (2.35). It will be shown in section 8 below how the domain of $\mathring{A}$ can be enlarged to obtain a selfadjoint positive extension A of $\mathring{A}$.

A Hilbert space $\mathcal{H}$ and selfadjoint positive operator A on $\mathcal{H}$ can be associated with each acoustic wave propagation problem by the method indicated above. A theory of solutions of the initial-boundary value problem (2.39) - (2.41) may then be based on A in the following way. First of all, the problem can be formulated as an initial value problem in $\mathcal{H}$. A function u: $R \to \mathcal{H}$ is sought such that

$$\frac{d^2u}{dt^2} + Au = 0 \text{ for all } t \in R \tag{2.50}$$

$$u(0) = f \text{ and } \frac{du(0)}{dt} = g \text{ in } \mathcal{H} \tag{2.51}$$

The spectral theorem for A:

$$A = \int_0^\infty \lambda \, d\Pi(\lambda) \tag{2.52}$$

and the associated operator calculus make it possible to construct the generalized solution

$$u(t) = (\cos t A^{1/2})f + (A^{-1/2} \sin t A^{1/2})g \tag{2.53}$$

The coefficient operators in (2.53) are bounded and hence u(t) is defined for all f and g in $\mathcal{H}$ and defines a curve in $C(R,\mathcal{H})$, the class of continuous $\mathcal{H}$-valued functions on R. The differentiability properties of u(t) depend on those of f and g. Two cases will be mentioned.

2.4 Solutions in $\mathcal{H}$

If $f \in \mathcal{H}$ and $g \in \mathcal{H}$ then u(t) is continuous in $\mathcal{H}$ and u(0) = f. However, u(t) will not in general be differentiable, and hence (2.50) and the second initial condition need not hold. In this case u(t) coincides with the "generalized solution in $\mathcal{H}$" which was defined and studied by M. Vishik and O. A. Ladyzhenskaya [32].

2.5 Solutions with finite energy

If $f \in D(A^{1/2})$ and $g \in \mathcal{H}$ then u is in the class

$$C^1(R,\mathcal{H}) \cap C(R,D(A^{1/2})) \tag{2.54}$$

This follows easily from (2.53) and the spectral theorem. Hence, u satisfies (2.51) but (2.50) need not hold. In this case u(t) coincides with the "solution with finite energy" which, for arbitrary domains Ω, was defined and studied by the author in [33,34, 42]. The existence and uniqueness of solutions with finite energy was proved in [33,34].

3. PROPAGATION IN HOMOGENEOUS FLUIDS

Propagation in an unlimited homogeneous fluid is analyzed in this section. In the notation of section 2 this is the special case where $\Omega = R^3$ and $\rho(x) = \rho$ and $c(x) = c$ are constant for all $x \in R^3$. It will be enough to treat the case $c = 1$ since the general case can be reduced to this one by the change of variable $ct \to t$. With these simplifications the wave equation (2.8) reduces to the d'Alembert equation

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) = 0 \tag{3.1}$$

and the propagation problem is simply the Cauchy problem for (3.1). The spectral and asymptotic analysis of solutions in $L_2(R^3)$ of (3.1) was developed in detail in [42]. Only the principal concepts and results are reviewed here.

The operator in $L_2(R^3)$ defined by $Au = -(\partial^2 u/\partial x_1^2 + \partial^2 u/\partial x_2^2 + \partial^2 u/\partial x_3^2)$ acting in the domain $D(A) = \mathcal{D}(R^3)$, the L. Schwartz space of testing functions, is known to be essentially selfadjoint [18]. Thus A has a unique selfadjoint extension in $L_2(R^3)$ which will be denoted here by A_0. This operator may be defined by

$$D(A_0) = L_2(R^3) \cap \left\{u:\ \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \in L_2(R^3)\right\} \tag{3.2}$$

and

$$A_0 u = -\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) \quad \text{for all } u \in D(A_0) \tag{3.3}$$

where the derivatives are interpreted in the sense of Schwartz's theory of distributions. A_0 is known to be non-negative and it is obviously real; that is

$$A_0\overline{u} = \overline{A_0 u} \quad \text{for all } u \in D(A_0) \tag{3.4}$$

where the bar denotes the complex conjugate.

The d'Alembert equation (3.1) will be interpreted as the equation

$$\frac{d^2u}{dt^2} + A_0 u = 0 \tag{3.5}$$

for an $L_2(R^3)$-valued function. Hence the solution in $L_2(R^3)$ of the Cauchy problem can be written

$$u(t) = (\cos t\, A_0^{1/2})f + (A_0^{-1/2}\sin t\, A_0^{1/2})g \tag{3.6}$$

where $u(0) = f$ and $du(0)/dt = g$ are in $L_2(R^3)$. If it is assumed that $f(x)$ and $g(x)$ are real-valued and

$$f \in L_2(R^3),\ g \in D(A_0^{-1/2}) \tag{3.7}$$

then it follows from (3.4) that

$$u(t,x) = \mathrm{Re}\, \{v(t,x)\} \tag{3.8}$$

where

$$v(t,\cdot) = \exp\, (-itA_0^{1/2})h \tag{3.9}$$

and

$$h = f + i\, A_0^{-1/2}\, g \in L_2(R^3) \tag{3.10}$$

In what follows attention is restricted to this case.

An eigenfunction expansion for A_0 may be based on the Plancherel theory of the Fourier transform in $L_2(R^3)$. If

$$w_0(x,p) = \frac{1}{(2\pi)^{3/2}} \exp\, (i\, x\cdot p),\ p \in R^3 \tag{3.11}$$

where $x \cdot p = x_1p_1 + x_2p_2 + x_3p_3$ then the main results of the theory state that for all $f \in L_2(R^3)$ the following limits exist

$$\left.\begin{aligned} \hat{f}(p) \equiv (\Phi_0 f)(p) &= L_2(R^3)\text{-}\lim_{M\to\infty} \int_{|x|\le M} \overline{w_0(x,p)}\, f(x)dx \\ f(x) = (\Phi_0^*\hat{f})(x) &= L_2(R^3)\text{-}\lim_{M\to\infty} \int_{|p|\le M} w_0(x,p)\, \hat{f}(p)dp \end{aligned}\right\} \tag{3.12}$$

and Φ_0: $L_2(R^3) \to L_2(R^3)$ is unitary. These relations will often be written in the symbolic form

$$\hat{f}(p) = \int_{R^3} \overline{w_0(x,p)}\, f(x)dx, \quad f(x) = \int_{R^3} w_0(x,p)\, \hat{f}(p)dp \quad (3.13)$$

but must be interpreted in the sense (3.12). The utility of the Fourier transform is due to the fact that if f and $\partial f/\partial x_j$ are in $L_2(R^3)$ then

$$\left(\Phi_0 \frac{\partial f}{\partial x_j}\right)(p) = ip_j\, \hat{f}(p),\ j = 1,2,3 \quad (3.14)$$

In particular, it follows that

$$D(A_0) = L_2(R^3) \cap \{u: \ |p|^2\hat{u}(p) \in L_2(R^3)\} \quad (3.15)$$

A_0 has the spectral representation

$$A_0 = \int_0^\infty \lambda\, d\Pi_0(\lambda) \quad (3.16)$$

with spectral family $\{\Pi_0(\lambda)\}$ defined by

$$\Pi_0(\lambda)f(x) = \int_{|p|\leq\sqrt{\lambda}} w_0(x,p)\, \hat{f}(p)dp,\ \lambda \geq 0 \quad (3.17)$$

It follows that A_0 is an absolutely continuous operator [18,42] whose spectrum is the interval $[0,\infty)$.

The above results imply that Φ_0 defines a spectral representation for A_0 and functions of A_0. In particular, if $\Psi(\lambda)$ is any bounded Lebesgue-measurable function of $\lambda \geq 0$ then

$$\Phi_0\Psi(A_0)f(p) = \Psi(|p|^2)\hat{f}(p) \quad (3.18)$$

These results imply that the wave function v(t,x) defined by (3.9) has the representation

$$v(t,x) = \int_{R^3} w_0(x,p)\, \exp\,(-it|p|)\, \hat{h}(p)dp \quad (3.19)$$

The function $w_0(x,p)$ is a generalized eigenfunction for A_0. This means that $w_0(\cdot,p)$ is locally in $D(A_0)$; i.e., $\phi w_0(\cdot,p) \in D(A_0)$ for every $\phi \in \mathcal{D}(R^3)$ and

$$A_0 w_0(\cdot,p) = \lambda\, w_0(\cdot,p),\ \lambda = |p|^2 \quad (3.20)$$

The functions

$$w_0(x,p)\ \exp\ (-it|p|) = \frac{1}{(2\pi)^{3/2}} \exp\ \{i(x\cdot p - t|p|)\} \tag{3.21}$$

are solutions of the d'Alembert equation which represent plane waves propagating in the direction of the vector $p \in R^3$. Hence, (3.19) is a representation of a localized acoustic wave as a superposition of the elementary waves (3.21).

The spectral integral (3.19) is the starting point for the asymptotic analysis of the behavior for $t \to \infty$ of solutions in $L_2(R^3)$ of the d'Alembert equation. It is convenient to begin the analysis with the special case where $\hat{h}$ is in the class

$$\mathcal{D}_0(R^3) = \mathcal{D}(R^3) \cap \{\hat{h}\colon\ \hat{h}(p) \equiv 0 \text{ for } |p| \le a,\ a = a(\hat{h}) > 0\} \tag{3.22}$$

The analysis will then be extended to the general case by using the easily verified fact that $\mathcal{D}_0(R^3)$ is dense in $L_2(R^3)$.

If $\hat{h} \in \mathcal{D}_0(R^3)$ and the support of $\hat{h}$ satisfies

$$\operatorname{supp} \hat{h} \subset \{p\colon\ 0 < a \le |p| \le b\} \tag{3.23}$$

then the spectral integral (3.19) converges both in $L_2(R^3)$ and pointwise to $v(t,x)$ and

$$v(t,x) = \frac{1}{(2\pi)^{3/2}} \int_{a\le|p|\le b} \exp\ \{i(x\cdot p - t|p|)\}\hat{h}(p)dp \tag{3.24}$$

To find the behavior of $v(t,\cdot) \in L_2(R^3)$ for $t \to \infty$ introduce spherical coordinates for p:

$$p = \rho\omega,\ \rho \ge 0,\ \omega \in S^2,\ dp = \rho^2 d\rho d\omega \tag{3.25}$$

where S^2 represents the unit sphere in R^3 with center at the origin and $d\omega$ is the element of area on S^2. This gives the representation

$$v(t,x) = \frac{1}{(2\pi)^{3/2}} \int_a^b e^{-it\rho}\ V(x,\rho)\ \rho^2 d\rho \tag{3.26}$$

where

$$V(x,\rho) = \int_{S^2} e^{i\rho x\cdot\omega}\ \hat{h}(\rho\omega)d\omega \tag{3.27}$$

The asymptotic behavior of $V(x,\rho)$ for $|x| \to \infty$ will be calculated and used to find the behavior of $v(t,x)$ for $t \to \infty$. Application of the method of stationary phase [2,23] to (3.27) with $x = r\theta$, $r \ge 0$, $\theta \in S^2$ implies that if

$$V(x,\rho) = \left(\frac{2\pi}{i\rho r}\right) e^{i\rho r}\, \hat{h}(\rho\theta) + \left(\frac{2\pi}{-i\rho r}\right) e^{-i\rho r}\, \hat{h}(-\rho\theta) + q_0(x,\rho) \tag{3.28}$$

then there exists a constant $M_0 = M_0(\hat{h})$ such that

$$|q_0(x,\rho)| \le M_0/r^2 \text{ for all } r > 0,\ a \le \rho \le b \text{ and } \theta \in S^2 \tag{3.29}$$

Substituting (3.28) into (3.26) gives

$$v(t,x) = G(r-t,\theta)/r + G'(r+t,\theta)/r + q_1(t,x) \tag{3.30}$$

where $G(\tau,\theta)$ and $G'(\tau,\theta)$ are the functions of $\tau \in R$ and $\theta \in S^2$ defined by

$$G(\tau,\theta) = \frac{1}{(2\pi)^{1/2}} \int_a^b e^{i\tau\rho}\, \hat{h}(\rho\theta)(-i\rho)\, d\rho \tag{3.31}$$

and

$$G'(\tau,\theta) = \frac{1}{(2\pi)^{1/2}} \int_{-b}^{-a} e^{i\tau\rho}\, \hat{h}(\rho\theta)(-i\rho)\, d\rho \tag{3.32}$$

Moreover, the estimate (3.29) implies that $q_1(t,x)$ satisfies

$$|q_1(t,x)| \le M_1/r^2 \text{ for all } r > 0,\ t \in R \text{ and } \theta \in S^2 \tag{3.33}$$

where $M_1 = M_1(\hat{h}) = (2\pi)^{-3/2}(b^3 - a^3)\, M_0(\hat{h})/3$.

The principal result of this section states that

$$v^{\infty}(t,x) = G(r-t,\theta)/r,\quad x = r\theta \tag{3.34}$$

is an asymptotic wave function for $v(t,x)$ in $L_2(R^3)$; that is,

$$\lim_{t\to\infty} \| v(t,\cdot) - v^{\infty}(t,\cdot)\|_{L_2(R^3)} = 0 \tag{3.35}$$

Before indicating a proof it is necessary to complete the statement of the theorem by defining the profile G for arbitrary $h \in L_2(R^3)$. When $\hat{h} \in \mathcal{D}_0(R^3)$, G is defined by (3.31) and a simple calculation gives

$$\|G\|^2_{L_2(R\times S^2)} = \int_{a\le|p|\le b} |\hat{h}(p)|^2 dp = \|\hat{h}\|^2_{L_2(R^3)} = \|h\|^2_{L_2(R^3)} \tag{3.36}$$

Hence the correspondence

$$h \to G = \Theta h \in L_2(R \times S^2) \tag{3.37}$$

can be extended to all $h \in L_2(R^3)$ by completion. Another method, based on the Plancherel theory in $L_2(R, L_2(S^2))$ is given in [42]. It is not difficult to verify by constructing Θ^{-1} that

$$\Theta:\ L_2(R^3) \to L_2(R \times S^2) \text{ is unitary} \tag{3.38}$$

A similar extension of the definition (3.32) of G' may be made.

A proof of (3.35) will now be outlined. Note first that the function $G'(r+t,\theta)/r$ tends to zero in $L_2(R^3)$ when $t \to \infty$. This follows from the simple calculation

$$\begin{aligned} \int_{R^3} |G'(r+t,\theta)/r|^2 dx &= \int_0^\infty \int_{S^2} |G'(r+t,\theta)|^2 d\theta dr \\ &= \int_t^\infty \int_{S^2} |G'(r,\theta)|^2 d\theta dr \end{aligned} \tag{3.39}$$

and the fact that $G' \in L_2(R \times S^2)$. The proof that, in (3.30), $q_1(t,\cdot) \to 0$ in $L_2(R^3)$ when $t \to \infty$ is based on the following lemma.

3.1 Convergence lemma

Let $\Omega \subset R^3$ be an unbounded domain and let $u(t,x)$ have the properties

$$u(t,\cdot) \in L_2(\Omega) \text{ for every } t > t_0 \tag{3.40}$$

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{L_2(K\cap\Omega)} = 0 \text{ for every compact } K \subset R^3 \tag{3.41}$$

$$|u(t,x)| \le M/|x|^2 \text{ for every } |x| > r_0 \tag{3.42}$$

where t_0, r_0 and M are constants. Then

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{L_2(\Omega)} = 0 \tag{3.43}$$

Only the case $\Omega = R^3$ of the lemma is needed here. The more general case is used in section 4. A simple proof of the lemma is given in [42].

The proof of (3.35) for the case $\hat{h} \in \mathcal{D}_0(R^3)$ may be completed by applying the lemma to $u(t,x) = q_1(t,x)$. (3.33) states that q_1

satisfies (3.42) while (3.40) and (3.41) follow from (3.30). To verify (3.41) note that $G'(r+t,\theta)/r$ satisfies it by (3.39). Moreover, if $K \subset \{x: |x| \le R\}$ then by direct calculation

$$\int_K |G(r-t,\theta)/r|^2 dx \le \int_{|x|\le R} |G(r-t,\theta)/r|^2 dx$$
$$= \int_0^R \int_{S^2} |G(r-t,\theta)|^2 d\theta dr = \int_{-t}^{R-t} \int_{S^2} |G(r,\theta)|^2 d\theta dr \tag{3.44}$$

The last integral tends to zero when $t \to \infty$ because $G \in L_2(R \times S^2)$. Finally, $v(t,x)$ satisfies (3.41). When $\hat{h} \in \mathcal{D}_0(R^3)$ this can be verified directly from (3.24) by an integration by parts.

The proof of (3.35) indicated above is valid when $\hat{h} \in \mathcal{D}_0(R^3)$. To prove (3.35) for general $h \in L_2(R^3)$ note that

$$v(t,\cdot) = U_0(t)h \text{ where } U_0(t) = \exp(-itA_0^{1/2}) \tag{3.45}$$

is unitary. In particular,

$$\|U_0(t)\| = 1 \text{ for all } t \in R \tag{3.46}$$

Similarly, if $U_0^\infty(t): L_2(R^3) \to L_2(R^3)$ is defined by

$$v^\infty(t,\cdot) = U_0^\infty(t)h \tag{3.47}$$

then it follows from (3.44) and (3.36) that $U_0^\infty(t)$ is contractive:

$$\|U_0^\infty(t)\| \le 1 \text{ for all } t \in R \tag{3.48}$$

The general case of (3.35) now follows from the special case $\hat{h} \in \mathcal{D}_0(R^3)$, the density of $\mathcal{D}_0(R^3)$ in $L_2(R^3)$ and the estimates (3.46) and (3.48). The details are given in [42].

The real part of the asymptotic wave function (3.34) is another function of the same form. Hence, (3.8) and (3.35) imply a similar result for the solution in $L_2(R^3)$ of the Cauchy problem. The result may be formulated as follows.

3.2 Theorem

Let f and g be real-valued functions such that $f \in L_2(R^3)$ and $g \in D(A_0^{-1/2})$. Let $u(t,x)$ be the corresponding solution in $L_2(R^3)$ of the d'Alembert equation given by (3.6). Define the asymptotic wave function

$$u^{\infty}(t,x) = \frac{F(r-t,\theta)}{r}, \quad x = r\theta \tag{3.49}$$

where

$$F(\tau,\theta) = \mathrm{Re}\,\{G(\tau,\theta)\} \tag{3.50}$$

and

$$G = \Theta h = \Theta(f + iA_0^{-1/2}g) \tag{3.51}$$

Then

$$\lim_{t\to\infty} \|u(t,\cdot) - u^{\infty}(t,\cdot)\|_{L_2(R^3)} = 0 \tag{3.52}$$

Stronger forms of convergence than (3.52) can also be proved under suitable hypotheses on the initial state. In particular, convergence in energy holds if the initial state has finite energy. A result of this type is formulated at the end of section 4 for the more general case of an initial-boundary value problem for the d'Alembert equation in an exterior domain.

4. SCATTERING BY OBSTACLES IN HOMOGENEOUS FLUIDS

The scattering of localized acoustic waves by bounded rigid obstacles immersed in an unlimited homogeneous fluid is analyzed in this section. The corresponding boundary value problem is

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) = 0 \text{ for } t > 0,\ x \in \Omega \tag{4.1}$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ for } t \geq 0,\ x \in \partial\Omega \tag{4.2}$$

$$u(0,x) = f(x) \text{ and } \partial u(0,x)/\partial t = g(x) \text{ for } x \in \Omega \tag{4.3}$$

where $\Omega \subset R^3$ is an exterior domain (i.e., $\Gamma = R^3 - \Omega$ is compact). This problem will be treated as a perturbation of the Cauchy problem of section 3.

A formulation of the initial-boundary value problem (4.1) - (4.3) which is applicable to arbitrary domains $\Omega \subset R^3$ was given by the author in [33,42]. That work provides the starting point for the analysis of this section and sections 5 and 6. The principal definitions and results are summarized here briefly.

The formulation makes use of the Hilbert space $L_2(\Omega)$ and the following subsets of $L_2(\Omega)$.

$$L_2^1(\Omega) = L_2(\Omega) \cap \{u: \ \partial u/\partial x_j \in L_2(\Omega) \text{ for } j = 1,2,3\} \tag{4.4}$$

$$L_2(\Delta,\Omega) = L_2(\Omega) \cap \{u: \ \Delta u \in L_2(\Omega)\} \tag{4.5}$$

$$L_2^1(\Delta,\Omega) = L_2^1(\Omega) \cap L_2(\Delta,\Omega) \tag{4.6}$$

where $\Delta u = \partial^2 u/\partial x_1^2 + \partial^2 u/\partial x_2^2 + \partial^2 u/\partial x_3^2$ denotes the Laplacian of u. The derivatives in these definitions are to be interpreted in the sense of the theory of distributions. The sets (4.4), (4.5) and (4.6) are linear subsets of $L_2(\Omega)$. Moreover, they are Hilbert spaces with inner products

$$(u,v)_1 = (u,v) + \sum_{j=1}^{3} (\partial u/\partial x_j, \partial v/\partial x_j) \tag{4.7}$$

$$(u,v)_\Delta = (u,v) + (\Delta u, \Delta v) \tag{4.8}$$

$$(u,v)_{1,\Delta} = (u,v)_1 + (\Delta u, \Delta v) \tag{4.9}$$

respectively, where (u,v) is the inner product in $L_2(\Omega)$.

4.1 Definition

A function $u \in L_2^1(\Delta,\Omega)$ is said to satisfy the generalized Neumann condition for Ω if and only if

$$(\Delta u, v) + \sum_{j=1}^{3} (\partial u/\partial x_j, \partial v/\partial x_j) = 0 \text{ for all } v \in L_2^1(\Omega) \tag{4.10}$$

Note that (4.10) defines a closed subspace

$$L_2^N(\Delta,\Omega) = L_2^1(\Delta,\Omega) \cap \{u: \ u \text{ satisfies } (4.10)\} \tag{4.11}$$

in the Hilbert space $L_2^1(\Delta,\Omega)$. The condition "$u \in L_2^N(\Delta,\Omega)$" is a generalization of the Neumann boundary condition (4.2). It is meaningful for arbitrary domains Ω. Moreover, it reduces to (4.2) whenever $\partial\Omega$ is sufficiently smooth (see [42,p.41] for a discussion).

The construction of solutions of the initial-boundary value problem (4.1) - (4.3) given below is based on the linear operator $A = A(\Omega)$ in $L_2(\Omega)$ defined by

$$D(A) = L_2^N(\Delta,\Omega) \tag{4.12}$$

$$Au = -\Delta u \text{ for all } u \in D(A) \tag{4.13}$$

The utility of this operator is based on the following theorem which is proved in [42].

4.2 Theorem

A is a selfadjoint real positive operator in $L_2(\Omega)$. Moreover, $D(A^{1/2}) = L_2^1(\Omega)$ and

$$\|A^{1/2}u\|^2 = \sum_{j=1}^{3} \|\partial u/\partial x_j\|^2 \text{ for all } u \in D(A^{1/2}) \quad (4.14)$$

The operator A may be used to construct "solutions in $L_2(\Omega)$" and "solutions with finite energy" of (4.1) - (4.3), as described in section 2. The solution in $L_2(\Omega)$ will be considered here. As in section 3, if $f \in L_2(\Omega)$ and $g \in D(A^{-1/2})$ then

$$u(t,x) = \mathrm{Re}\,\{v(t,x)\} \quad (4.15)$$

where

$$v(t,\cdot) = \exp\,(-itA^{1/2})h, \quad h = f + iA^{-1/2}g \quad (4.16)$$

The properties of the operator A stated in the theorem above are valid for arbitrary domains $\Omega \subset R^3$. It was shown in [42] that if Ω is an exterior domain then A has a continuous spectrum. Moreover, if Ω has the local compactness property (defined below) then there exist eigenfunction expansions for A in terms of generalized eigenfunctions which are perturbations of the plane wave eigenfunctions of section 3. In the remainder of this section the eigenfunction expansions are described and used to analyze the structure of solutions of the scattering problem (4.1) - (4.3). The principal result of the analysis states that the behavior of the acoustic field for large times is described by an asymptotic wave function of exactly the same form (3.49) as when there is no obstacle. The only effect of an obstacle is to modify the wave profile $F(\tau,\theta)$. Moreover, a procedure is given for calculating the modified profile when the obstacle and the initial state are known.

4.3 Distorted plane wave eigenfunctions

Two families of generalized eigenfunctions of A, denoted by $w_+(x,p)$ and $w_-(x,p)$ respectively, were defined in [42]. They are perturbations of the plane wave eigenfunctions $w_0(x,p)$ and have the form

$$w_\pm(x,p) = w_0(x,p) + w_\pm'(x,p), \quad p \in R^3 \quad (4.17)$$

where $w_+'(x,p)$ and $w_-'(x,p)$ may be interpreted as secondary fields which are produced when the obstacle $\Gamma = R^3 - \Omega$ is irradiated by the plane wave $w_0(x,p)$. Mathematically, $w_+(x,p)$ and $w_-(x,p)$ must satisfy

$$(\Delta + |p|^2)\ w_{\pm}(x,p) = 0 \text{ for } x \in \Omega \tag{4.18}$$

$$\frac{\partial w_{\pm}(x,p)}{\partial \nu} = 0 \text{ for } x \in \partial\Omega \tag{4.19}$$

However, they are not completely determined by these conditions. Instead, $w_+(x,p)$ is determined by (4.18), (4.19) and the condition that $w_+'(x,p)$ should describe an outgoing secondary wave. This is implied by the Sommerfeld condition for outgoing waves:

$$\left.\begin{aligned} &\frac{\partial w_+'(x,p)}{\partial |x|} - i|p|\ w_+'(x,p) = o(|x|^{-1}),\quad |x| \to \infty \\ &w_+'(x,p) = O(|x|^{-1}),\quad |x| \to \infty \end{aligned}\right\} \tag{4.20}$$

Similarly, $w_-(x,p)$ is determined by (4.18), (4.19) and the condition that $w_-'(x,p)$ should describe an incoming secondary wave, which is implied by the Sommerfeld condition for incoming waves:

$$\left.\begin{aligned} &\frac{\partial w_-'(x,p)}{\partial |x|} + i|p|\ w_-'(x,p) = o(|x|^{-1}),\quad |x| \to \infty \\ &w_-'(x,p) = O(|x|^{-1}),\quad |x| \to \infty \end{aligned}\right\} \tag{4.21}$$

Of course, if $\partial\Omega$ is not smooth then the boundary condition (4.19) must be understood in the generalized sense of (4.10). A technical difficulty is caused by the fact that $w_{\pm}(\cdot,p)$ cannot be in $D(A) = L_2^N(\Delta,\Omega)$ because the spectrum of A is continuous. This is overcome by requiring that

$$\phi w_{\pm}(\cdot,p) \in L_2^N(\Delta,\Omega) \tag{4.22}$$

for all $\phi \in \mathcal{D}(R^3)$ such that $\phi(x) \equiv 1$ in a neighborhood of $\partial\Omega$. Generalized eigenfunctions with these properties will be called "distorted plane waves," following T. Ikebe [16].

The uniqueness of distorted plane waves satisfying (4.18), (4.20) or (4.21) and (4.22) was proved in [42] for arbitrary exterior domains. However, to prove their existence it was necessary to impose a condition on $\partial\Omega$. To define it let

$$\Omega_R = \Omega \cap \{x:\ |x| < R\} \tag{4.23}$$

$$L_2^{\ell oc}(\bar{\Omega}) = \{u:\ u \in L_2(\Omega_R) \text{ for every } R > 0\} \tag{4.24}$$

$$L_2^{1,\ell oc}(\bar{\Omega}) = L_2^{\ell oc}(\bar{\Omega}) \cap \{u: \partial u/\partial x_j \in L_2^{\ell oc}(\bar{\Omega}) \quad \text{for } j = 1,2,3\} \tag{4.25}$$

and define the

4.4 Local compactness property

A domain $\Omega \subset R^3$ is said to have the local compactness property if and only if for each set $S \subset L^{1,\ell oc}(\bar{\Omega})$ and each $R > 0$ the condition

$$\|u\|_{L_2^1(\Omega_R)} \leq C(R) \text{ for all } u \in S \tag{4.26}$$

implies that S is precompact in $L_2(\Omega_R)$; i.e., every sequence $\{u_n\}$ in S which satisfies (4.26) has a subsequence which converges in $L_2(\Omega_R)$. The class of domains with the local compactness property will be denoted by LC.

The local compactness property is known to hold for large classes of domains. S. Agmon has proved it for domains with the "segment property" [1]. A generalization of the segment property, called the "finite tiling property" was given by the author in [42]. As an application of this condition it can be shown that the local compactness property holds for the many simple, but non-smooth, boundaries that arise in applications, such as polyhedra, finite sections of cylinders, cones, spheres, disks, etc. The following existence theorem was proved in [42].

4.5 Theorem

Let $\Omega \subset R^3$ be an exterior domain such that $\Omega \in LC$. Then for each $p \in R^3$ there exists a unique outgoing distorted plane wave $w_+(x,p)$ and a unique incoming distorted plane wave $w_-(x,p)$.

The outgoing (resp. incoming) property of $w_+'(x,p)$ (resp. $w_-'(x,p)$) is made explicit by the following corollary.

4.6 Corollary

Under the same hypotheses there exist functions $T_\pm(\theta,p) \in C^\infty(S^2 \times \{R^3 - 0\})$ such that

$$w_\pm'(x,p) = \frac{e^{\pm i|p|r}}{r} T_\pm(\theta,p) + w_\pm''(x,p), \quad x = r\theta \tag{4.27}$$

where

$$w''_{\pm}(x,p) = O(r^{-2}), \quad r \to \infty \tag{4.28}$$

uniformly for $\theta = x/r \in S^2$ and p in any compact subset of $R^3 - \{0\}$.

In acoustics the functions $T_+(\theta,p)$ and $T_-(\theta,p)$ are called the far-field amplitudes of the distorted plane waves.

4.7 The eigenfunction expansion theorem

Each of the families $\{w_+(\cdot,p): \; p \in R^3\}$ and $\{w_-(\cdot,p): \; p \in R^3\}$ defines a complete set of generalized eigenfunctions of A in the sense described by the following theorems.

4.8 Theorem

For each $f \in L_2(\Omega)$ the following limits exist

$$\left.\begin{aligned} \hat{f}_{\pm}(p) &= L_2(R^3)\text{-}\lim_{M\to\infty} \int_{\Omega_M} \overline{w_{\pm}(x,p)}\, f(x)\,dx \\ f(x) &= L_2(\Omega)\text{-}\lim_{M\to\infty} \int_{|p|\le M} w_{\pm}(x,p)\, \hat{f}_{\pm}(p)\,dp \end{aligned}\right\} \tag{4.29}$$

where $\Omega_M = \Omega \cap \{x: \; |x| < M\}$. Moreover, the operators $\Phi_{\pm}: \; L_2(\Omega) \to L_2(R^3)$ defined by

$$\Phi_{\pm} f = \hat{f}_{\pm} \tag{4.30}$$

are unitary.

The relations (4.29) will usually be written in the symbolic form

$$\hat{f}_{\pm}(p) = \int_{\Omega} \overline{w_{\pm}(x,p)}\, f(x)\,dx, \quad f(x) = \int_{R^3} w_{\pm}(x,p)\, \hat{f}_{\pm}(p)\,dp \tag{4.31}$$

but must be understood in the sense of (4.29).

4.9 Theorem

If $\{\Pi(\lambda)\}$ denotes the spectral family of A:

$$A = \int_0^{\infty} \lambda \; d\Pi(\lambda) \tag{4.32}$$

then $\Pi(\lambda)$ has the eigenfunction expansions

$$\Pi(\lambda)\, f(x) = \int_{|p|\leq\sqrt{\lambda}} w_{\pm}(x,p)\, \hat{f}_{\pm}(p)dp, \quad \lambda \geq 0 \tag{4.33}$$

In particular, A is an absolutely continuous operator whose spectrum is the interval $[0,\infty)$.

The last result implies that Φ_+ and Φ_- define spectral representations for A in the sense of the following corollary.

4.10 Corollary

If $\Psi(\lambda)$ is a bounded Lebesgue-measurable function of $\lambda \geq 0$ then for all $f \in L_2(\Omega)$

$$\Phi_{\pm}\Psi(A)f(p) = \Psi(|p|^2)\, \hat{f}_{\pm}(p) \tag{4.34}$$

These results provide a complete generalization of the Plancherel theory to exterior domains $\Omega \in LC$.

4.11 The eigenfunction expansions and scattering theory

The results stated above imply that the wave functions

$$v(t,\cdot) = \exp\,(-itA^{1/2})h, \quad h \in L_2(\Omega) \tag{4.35}$$

have the spectral integral representations

$$v(t,x) = \int_{R^3} w_{\pm}(x,p)\, \exp\,(-it|p|)\, \hat{h}_{\pm}(p)dp \tag{4.36}$$

Note that (4.36) defines <u>two</u> representations, corresponding to $w_+(x,p)$ and $w_-(x,p)$. They will be called the outgoing and incoming representations, respectively.

The representations (4.36) and the results of section 3 will now be used to derive the asymptotic behavior of $v(t,x)$ for $t \to \infty$. To begin consider an initial state $h \in L_2(\Omega)$ such that

$$\hat{h}_- \in \mathcal{D}_0(R^3) \tag{4.37}$$

Such states are dense in $L_2(\Omega)$ because $\mathcal{D}_0(R^3)$ is dense in $L_2(R^3)$ and $\Phi_-\colon\ L_2(\Omega) \to L_2(R^3)$ is unitary. The wave function corresponding to (4.37) is

$$v(t,x) = \int_{R^3} w_-(x,p) \exp(-it|p|) \hat{h}_-(p)dp \tag{4.38}$$

where the integral converges both pointwise and in $L_2(\Omega)$ to $v(t,x)$. To discover the behavior of $v(t,x)$ for $t \to \infty$ substitute the decompositions (4.17) and (4.27) for $w_-(x,p)$ into (4.38) and write

$$v(t,x) = v_0(t,x) + v'(t,x) + v''(t,x) \tag{4.39}$$

where

$$v_0(t,x) = \int_{R^3} w_0(x,p) \exp(-it|p|) \hat{h}_-(p)dp \tag{4.40}$$

$$v'(t,x) = \frac{1}{r} \int_{R^3} \exp\{-i|p|(r+t)\} T_-(\theta,p) \hat{h}_-(p)dp \tag{4.41}$$

and

$$v''(t,x) = \int_{R^3} w''_-(x,p) \exp(-it|p|) \hat{h}_-(p)dp \tag{4.42}$$

Note that $v_0(t,x)$ is a solution in $L_2(R^3)$ of the d'Alembert equation. Indeed, $\hat{h}_- = \Phi_- h = \Phi_0(\Phi_0^* \Phi_- h) = \hat{h}_0$ where

$$h_0 = \Phi_0^* \Phi_- h \in L_2(R^3) \tag{4.43}$$

and

$$\begin{aligned} v_0(t,\cdot) &= \int_{R^3} w_0(\cdot,p) \exp(-it|p|) \hat{h}_0(p)dp \\ &= \exp(-itA_0^{1/2})h_0 \end{aligned} \tag{4.44}$$

Thus $v_0(t,x)$ represents a wave in an unlimited fluid containing no obstacles. It will be shown that $v(t,x)$ is asymptotically equal to this wave when $t \to \infty$; i.e.,

$$\lim_{t\to\infty} \| v(t,\cdot) - v_0(t,\cdot)\|_{L_2(\Omega)} = 0 \tag{4.45}$$

To see this note that, in (4.39), $v'(t,x)$ has the form

$$v'(t,x) = G'(r+t,\theta)/r \tag{4.46}$$

It was shown in section 3 that such functions tend to zero in $L_2(R^3)$ when $t \to \infty$ (see (3.39)). It is easy to check that (4.37) implies that $G' \in L_2(R \times S^2)$. Finally, condition (4.28) for $w''_-(x,p)$ implies that the term $v''(t,x)$ in (4.39) satisfies

$$|v''(t,x)| \leq M/|x|^2 \text{ for all } |x| > 0 \text{ and } t \in R \qquad (4.47)$$

with a suitable constant M. Hence, the convergence lemma of section 3, applied to $v'' = v - v_0 - v'$ implies (4.45) if $v''(t,x)$ satisfies the local decay condition (3.41). For $v'(t,x)$ this condition follows from (4.46). For $v(t,x)$ and $v_0(t,x)$ it follows from the local compactness property. A proof may be found in [42]. Thus (4.45) is established for all $\hat{h}_- \in \mathcal{D}_0(R^3)$. The main result of this section is the

4.12 Theorem

For all $h \in L_2(\Omega)$ if $v(t,\cdot) = \exp(-itA^{1/2})h$ and $v_0(t,\cdot) = \exp(-itA_0^{1/2})(\Phi_0^*\Phi_-)h$ then

$$\lim_{t\to\infty} \| v(t,\cdot) - v_0(t,\cdot)\|_{L_2(\Omega)} = 0 \qquad (4.48)$$

This result follows immediately from the special case (4.37) proved above, the density of $\mathcal{D}_0(R^3)$ in $L_2(R^3)$ and the unitarity of the operators $\exp(-itA^{1/2})$, $\exp(-itA_0^{1/2})$, Φ_0 and Φ_-.

4.13 Corollary

If J_Ω: $L_2(\Omega) \to L_2(R^3)$ is defined by $J_\Omega u(x) = u(x)$ for all $x \in \Omega$ and $J_\Omega u(x) = 0$ for all $x \in R^3 - \Omega$ then the strong limit

$$W_+ = W_+(A_0^{1/2}, A^{1/2}, J_\Omega) = \operatorname{s-lim}_{t\to\infty} \exp(itA_0^{1/2}) J_\Omega \exp(-itA^{1/2}) \qquad (4.49)$$

exists in $L_2(\Omega)$ and W_+: $L_2(\Omega) \to L_2(R^3)$ is given by

$$W_+ = \Phi_0^*\Phi_- \qquad (4.50)$$

In particular, W_+ is unitary.

The operator W_+ is the wave operator for the pair $A_0^{1/2}$, $A^{1/2}$ in the sense of the time dependent theory of scattering. The equivalence of (4.48) and (4.50) is proved in [42].

4.14 Asymptotic wave functions in $L_2(\Omega)$

The wave function in $L_2(R^3)$ defined by

$$v_0(t,\cdot) = \exp(-itA_0^{1/2})h_0, \quad h_0 = \Phi_0^*\Phi_- h \qquad (4.51)$$

has an asymptotic wave function in $L_2(R^3)$, by the results of section 3; i.e.,

$$\lim_{t\to\infty} \| v_0(t,\cdot) - v^\infty(t,\cdot)\|_{L_2(R^3)} = 0 \tag{4.52}$$

where

$$v^\infty(t,x) = G(r-t,\theta)/r, \quad x = r\theta \tag{4.53}$$

and

$$G = \Theta h_0 = \Theta\Phi_0^* \Phi_- h \tag{4.54}$$

Equations (4.48), (4.52) and the triangle inequality imply the

4.15 Theorem

For each $h \in L_2(\Omega)$ the wave function $v^\infty(t,\cdot)$ defined by (4.53), (4.54) is an asymptotic wave function in $L_2(\Omega)$ for $v(t,\cdot) = \exp(-itA^{1/2})h$; that is,

$$\lim_{t\to\infty} \| v(t,\cdot) - v^\infty(t,\cdot)\|_{L_2(\Omega)} = 0 \tag{4.55}$$

4.16 Corollary

The profile of the asymptotic wave function is given by

$$G(\tau,\theta) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{i\tau\rho}\, \hat{h}_-(\rho\theta)(-i\rho)\,d\rho \tag{4.56}$$

where the integral converges in $L_2(R \times S^2)$.

This follows immediately from (4.54) and (3.31). Note that the only difference between the asymptotic wave functions for R^3 and those for Ω is that $\hat{h} = \Phi_0 h$ is replaced by $\hat{h}_- = \Phi_- h$.

4.17 Asymptotic energy distributions

If the initial state $h \in L_2(\Omega)$ has derivatives in $L_2(\Omega)$ then the corresponding profile G and asymptotic wave function $v^\infty(t,x)$ will have corresponding derivatives. In particular, the following result was proved in [42].

4.18 Corollary

If $\partial h(x)/\partial x_j \in L_2(\Omega)$ for $j = 1,2,3$ then $\partial v(t,x)/\partial t$ and $\partial v(t,x)/\partial x_j$ are in $L_2(\Omega)$ for all $t \in R$ and $j = 1,2,3$ and

$$\left.\begin{aligned} &\lim_{t\to\infty} \|\partial v(t,\cdot)/\partial t - v_0^\infty(t,\cdot)\|_{L_2(\Omega)} = 0 \\ &\lim_{t\to\infty} \|\partial v(t,\cdot)/\partial x_j - v_j^\infty(t,\cdot)\|_{L_2(\Omega)} = 0,\ j = 1,2,3 \end{aligned}\right\} \tag{4.57}$$

where

$$v_k^\infty(t,x) = G_k(r - t,\theta)/r,\ k = 0,1,2,3 \tag{4.58}$$

$$G_0(\tau,\theta) = -\partial G(\tau,\theta)/\partial\tau \tag{4.59}$$

$$G_j(\tau,\theta) = -G_0(\tau,\theta)\theta_j,\ j = 1,2,3 \tag{4.60}$$

and $G(\tau,\theta)$ is given by (4.56).

The energy integral for a homogeneous fluid is given by

$$E(u,K,t) = \frac{1}{2}\int_K \left\{\left(\frac{\partial u(t,x)}{\partial t}\right)^2 + \sum_{j=1}^{3}\left(\frac{\partial u(t,x)}{\partial x_j}\right)^2\right\} dx \tag{4.61}$$

if $\rho = 1$, $c = 1$. The last corollary implies that if $u(t,x) = \mathrm{Re}\,\{v(t,x)\}$ is a solution with finite energy in Ω then the energy in any measurable cone

$$C = \{x = r\theta:\ r > 0,\ \theta \in C_0 \subset S^2\} \tag{4.62}$$

has a limit as $t \to \infty$ which can be calculated from the initial state $u(0,x) = f(x)$, $\partial u(0,x)/\partial t = g(x)$. The following result was proved in [42].

4.19 Theorem

If $f \in L_2^1(\Omega)$, $g \in L_2(\Omega)$ and if C is any measurable cone in $\mathbf{R}^3$ then

$$\lim_{t\to\infty} E(u,C \cap \Omega,t) = \frac{1}{2}\int_C \left||p|\hat{f}_-(p) + i\hat{g}_-(p)\right|^2 dp \tag{4.63}$$

5. PROPAGATION IN UNIFORM TUBULAR WAVEGUIDES

The propagation and scattering of localized acoustic waves is simple and compound tubular waveguides with rigid walls, and filled with a homogeneous fluid, is analyzed in this section and the next. The simplest case is the uniform semi-infinite cylinder, closed by a plane wall perpendicular to the axis. Other special

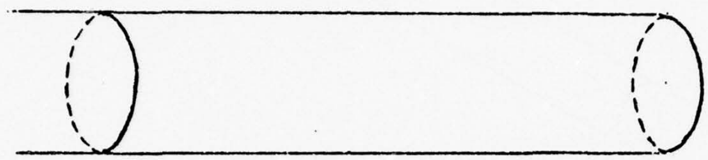

Figure 1. Uniform semi-infinite cylindrical waveguide.

cases which are of interest in applied acoustics include the cylindrical waveguide terminated by a resonator, the tubular

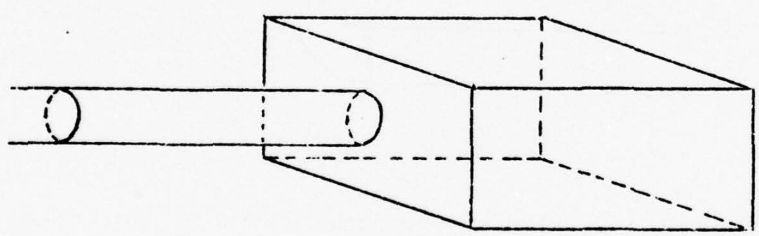

Figure 2. Cylindrical waveguide terminated by a resonator.

waveguide with a bend, or elbow, coupled cylindrical waveguides with different cross-sections, the T-joint in a waveguide, uniform waveguides containing an iris, waveguides containing obstacles, and many others.

The most general compound tubular waveguide considered here is described by a domain $\Omega \subset R^3$ of the form

$$\Omega = \Omega_0 \cup S_1 \cup S_2 \cup \cdots \cup S_m \qquad (5.1)$$

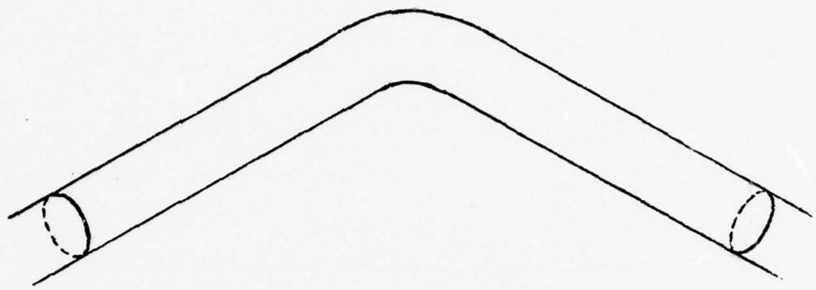

Figure 3. Waveguide with elbow.

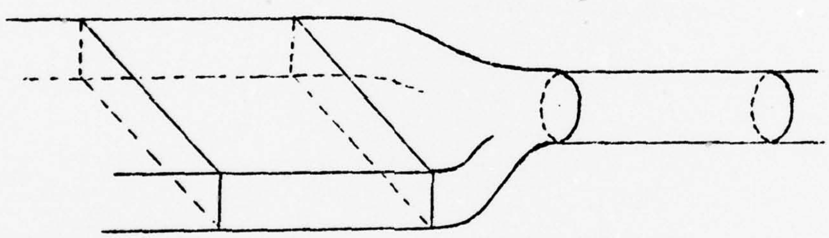

Figure 4. Coupled waveguides.

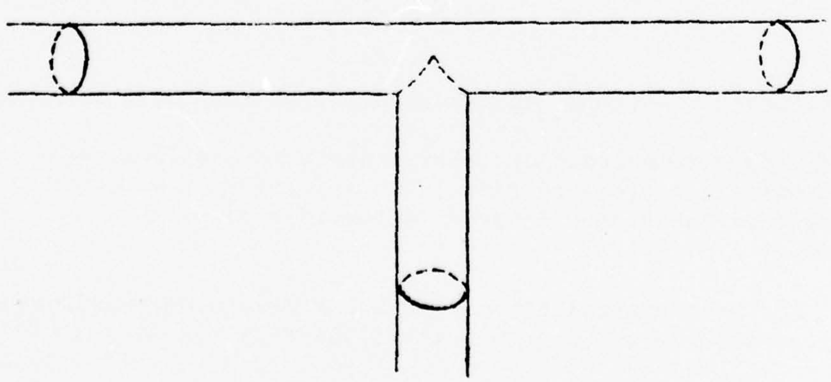

Figure 5. Waveguide with T-joint.

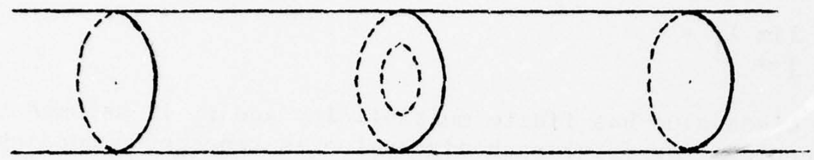

Figure 6. Waveguide with iris.

where Ω_0 is a bounded domain and $S_1, S_2, \cdots, S_m$ are disjoint uniform semi-infinite cylinders. If Ω is a waveguide with rigid walls, filled with a homogeneous fluid, the corresponding boundary value problem is again problem (4.1) - (4.3), but for a domain with the structure (5.1). Hence, the Hilbert space formulation of (4.1) - (4.3) given at the beginning of section 4, which is valid for arbitrary domains $\Omega \subset R^3$, provides a starting point for the analysis of the waveguide problems. The remainder of this section presents the spectral and asymptotic analysis of acoustic waves in a uniform semi-infinite cylindrical waveguide. The general case (5.1) is analyzed in section 6.

5.1 The uniform semi-infinite cylinder

It will be convenient to use coordinates

$$(x_1, x_2, y) \equiv (x, y) \in R^3 \tag{5.2}$$

such that the y-axis lies in the waveguide. With this choice the waveguide may be described by a domain of the form

$$S = \{(x,y): \; x \in G \text{ and } y > 0\} \tag{5.3}$$

where $G \subset R^2$ defines the waveguide cross section. It will be assumed that G is bounded and that $S \in LC$.

The spectral analysis of the operator $A = A(S)$, acting in $L_2(S)$, will be based on the spectral analysis of $A(G)$ acting in $L_2(G)$. It can be shown that the hypothesis $S \in LC$ implies that $G \in LC$ as a domain in R^2. This property and the boundedness of

G imply that $A(G)$ has a discrete spectrum with eigenvalues

$$0 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \tag{5.4}$$

such that

$$\lim_{j\to\infty} \lambda_j = \infty \tag{5.5}$$

Each eigenvalue has finite multiplicity and it is assumed that in the enumeration (5.4) each eigenvalue is repeated according to its multiplicity. There exists a corresponding orthonormal set $\{\phi_j(x)\}$ of eigenfunctions which is complete in $L_2(G)$. Each ϕ_j satisfies $\phi_j \in D(A(G)) = L_2^N(\Delta, G)$ and $A(G)\phi_j = \lambda_j\phi_j$. Formally, the $\phi_j(x)$ are solutions of the eigenvalue problem

$$\left.\begin{aligned} &\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \lambda\phi = 0 \text{ for } x \in G \\ &\frac{\partial\phi}{\partial\nu} = 0 \text{ for } x \in \partial G \end{aligned}\right\} \tag{5.6}$$

Of course, if ∂G is not smooth then the boundary condition is the generalized Neumann condition defined in section 4. It is known that the first eigenvalue $\lambda_0 = 0$ is simple with normalized eigenfunction

$$\phi_0(x) = \frac{1}{|G|^{1/2}} = \text{const.} \tag{5.7}$$

where $|G|$ is the Lebesgue measure of G.

5.2 The eigenfunction expansion

The eigenfunctions of A may be constructed by separation of variables. From a more sophisticated point of view, A is a sum of tensor products

$$A = A(G) \otimes 1 + 1 \otimes A(R_+) \tag{5.8}$$

where $R_+ = \{y: \ y > 0\}$. It follows that the eigenfunctions of A are products of eigenfunctions of $A(G)$ and $A(R_+)$. The spectral analysis of $A(R_+)$ is given by the Fourier cosine transform in $L_2(R_+)$:

$$\hat{f}(p) = L_2(R_+)\text{-}\lim_{M\to\infty} \left(\frac{2}{\pi}\right)^{1/2} \int_0^M \cos py\, f(y)dy \tag{5.9}$$

$$f(y) = L_2(R_+)\text{-}\lim_{M\to\infty} \left(\frac{2}{\pi}\right)^{1/2} \int_0^M \cos py\, \hat{f}(p)dp \tag{5.10}$$

$$\|\hat{f}\|_{L_2(R_+)} = \|f\|_{L_2(R_+)} \tag{5.11}$$

It follows that a complete normalized family of generalized eigenfunctions for A is defined by

$$w_j(x,y,p) = \left(\frac{2}{\pi}\right)^{1/2} \cos py\ \phi_j(x),\ p \in R_+,\ j = 0,1,2,\cdots \tag{5.12}$$

The Plancherel theory for $A(G)$ and $A(R_+)$, quoted above, implies that

$$\hat{f}_j(p) = L_2(R_+)\text{-}\lim_{M\to\infty} \int_0^M \int_G \overline{w_j(x,y,p)}\ f(x,y)dxdy \tag{5.13}$$

exists for all $f \in L_2(S)$, and the operator Φ_j: $L_2(S) \to L_2(R_+)$ defined by $\Phi_j f = \hat{f}_j$ has range $\Phi_j L_2(S) = L_2(R_+)$. Moreover

$$\|f\|^2_{L_2(S)} = \sum_{j=0}^{\infty} \|\hat{f}_j\|^2_{L_2(R_+)} \tag{5.14}$$

and

$$f(x,y) = L_2(S)\text{-}\lim_{M,N\to\infty} \sum_{j=0}^{N} \int_0^M w_j(x,y,p)\ \hat{f}_j(p)dp \tag{5.15}$$

The relations (5.13) and (5.15) are frequently written in the more concise symbolic form

$$\hat{f}_j(p) = \int_S \overline{w_j(x,y,p)}\ f(x,y)dxdy \tag{5.16}$$

and

$$f(x,y) = \sum_{j=0}^{\infty} \int_{R_+} w_j(x,y,p)\ \hat{f}_j(p)dp \tag{5.17}$$

but must be understood in the sense of (5.13) and (5.15).

Note that, formally, $\hat{f}_j(p)$ is just the $L_2(S)$ inner product of $f(x,y)$ and the eigenfunction (5.12). For a more detailed discussion of this expansion see [21].

The generalized eigenfunctions (5.12) are locally in $D(A)$ and satisfy

$$Aw_j(\cdot,\cdot,p) \equiv -\Delta w_j(\cdot,\cdot,p) = (p^2 + \lambda_j) w_j(\cdot,\cdot,p) \tag{5.18}$$

This fact and the Plancherel theory imply the following construction of the spectral family of A.

5.3 Theorem

If $\{\Pi(\lambda), \lambda \geq 0\}$ denotes the spectral family of $A = A(S)$ then $\Pi(\lambda)$ has the eigenfunction expansion

$$\begin{aligned} \Pi(\lambda)\ f(x,y) &= \sum_{\lambda_j \leq \lambda} \int_0^{\sqrt{\lambda-\lambda_j}} w_j(x,y,p)\ \hat{f}_j(p)dp \\ &= \sum_{\lambda_j \leq \lambda} \left\{ \int_0^{\sqrt{\lambda-\lambda_j}} \left(\frac{2}{\pi}\right)^{1/2} \cos py\ \hat{f}_j(p)dp \right\} \phi_j(x) \end{aligned} \tag{5.19}$$

for all $\lambda \geq 0$. In particular, A is an absolutely continuous operator whose spectrum is the interval $[\lambda_0,\infty) = [0,\infty)$.

Note that the sum in (5.19) is actually finite by (5.5). (5.19) implies that the eigenfunction expansion (5.17) defines a spectral representation for A in the sense of the following corollary.

5.4 Corollary

If $\Psi(\lambda)$ is any bounded Lebesgue-measurable function of $\lambda \geq 0$ then for all $f \in L_2(S)$

$$\Psi(A)\ f(x,y) = L_2(S)\text{-}\lim_{M,N\to\infty} \sum_{j=0}^{N} \int_0^M w_j(x,y,p)\ \Psi(p^2+\lambda_j)\ \hat{f}_j(p)dp \tag{5.20}$$

The eigenfunction expansion (5.17) defines a decomposition of the Hilbert space $L_2(S)$. To describe its properties let $f \in L_2(S)$ and define

$$\begin{aligned} P_j\ f(x,y) &= \left\{ \int_G \overline{\phi_j(x')}\ f(x',y)dx' \right\} \phi_j(x) \\ &= f_j(y)\ \phi_j(x),\ j = 0,1,2,\cdots \end{aligned} \tag{5.21}$$

where

$$f_j(y) = \int_G \overline{\phi_j(x')}\, f(x',y)dx', \ j = 0,1,2,\cdots \tag{5.22}$$

The orthonormality of $\{\phi_j\}$ in $L_2(G)$ implies that $\{P_j: j = 0,1,2,\cdots\}$ defines a complete family of orthogonal projections in $L_2(S)$:

$$P_j^* = P_j, \ P_j P_k = \delta_{jk} P_k \text{ for } j,k = 0,1,2,\cdots \tag{5.23}$$

and

$$\sum_{j=0}^{\infty} P_j = 1 \tag{5.24}$$

Moreover, a simple calculation gives

$$\Psi(A)\ P_j f = P_j \Psi(A) f = \int_{R_+} w_j(\cdot,\cdot,p)\Psi(p^2 + \lambda_j)\hat{f}_j(p)dp \tag{5.25}$$

for $j = 0,1,2,\cdots$. In particular,

$$P_j f(x,y) = \int_{R_+} w_j(x,y,p)\ \hat{f}_j(p)dp \tag{5.26}$$

An equivalent operator-theoretic representation is $P_j = \Phi_j^* \Phi_j$. If

$$\mathcal{H}_j = P_j L_2(S) = \{f(x,y) = f_j(y)\phi_j(x): \ f_j \in L_2(R_+)\} \tag{5.27}$$

then (5.23) - (5.25) imply the

5.5 Corollary

The direct sum decomposition

$$L_2(S) = \sum_{j=0}^{\infty} \oplus \mathcal{H}_j \tag{5.28}$$

is a reducing decomposition for A.

Note that each $\mathcal{H}_j$ is isomorphic to $L_2(R_+)$ under the mapping $f(x,y) \to f_j(y)$ defined by (5.22).

5.6 Solutions in $L_2(S)$ of the propagation problem

Only the case where $f \in L_2(S)$ and $g \in D(A^{-1/2})$ will be discussed. As in sections 3 and 4, the solution in $L_2(S)$ of the propagation problem (4.1) - (4.3) has the form

$$u(t,x,y) = \text{Re}\,\{v(t,x,y)\} \tag{5.29}$$

where

$$v(t,\cdot,\cdot) = \exp\,(-it\,A^{1/2})h, \quad h = f + iA^{-1/2}g \in L_2(S) \tag{5.30}$$

The decomposition (5.28) implies that

$$v(t,x,y) = \sum_{j=0}^{\infty} v_j(t,x,y) \text{ in } L_2(S) \tag{5.31}$$

where

$$v_j(t,x,y) = P_j v(t,x,y) = v_j(t,y)\phi_j(x) \in L_2(S) \tag{5.32}$$

with

$$v_j(t,y) = \left(\frac{2}{\pi}\right)^{1/2} \int_{R_+} \cos py\; e^{-it\omega_j(p)}\, \hat{h}_j(p)dp \tag{5.33}$$

and

$$\omega_j(p) = (p^2 + \lambda_j)^{1/2} \geq \lambda_j^{1/2} \geq 0 \tag{5.34}$$

In the theory of waveguides (5.31) is called a modal decomposition and the partial waves $v_j(t,x,y)$ are called waveguide modes. $v_j(t,x,y)$ will be said to be in mode j of the waveguide S. In particular, mode 0

$$v_0(t,x,y) = v_0(t,y)/|G|^{1/2} \tag{5.35}$$

will be called the fundamental mode of S. It is not difficult to show that

$$\begin{aligned} u_0(t,y) &= \text{Re}\,\{v_0(t,y)\} \\ &= \frac{1}{2}\,\{f_0(y-t) + f_0(y+t)\} + \frac{1}{2}\int_{y-t}^{y+t} g_0(y')dy' \end{aligned} \tag{5.36}$$

where $f_0(-y) = f_0(y)$ and $g_0(-y) = g_0(y)$. Note that the modal waves propagate independently in the sense that different modes are orthogonal in $L_2(S)$ for all t.

The spectral representation (5.31), (5.32), (5.33) will now be used to study the asymptotic behavior for $t \to \infty$ of solutions in $L_2(S)$. Because of the independence of the modes it will be enough to study the individual modal waves (5.33). The substitution $2 \cos py = \exp(ipy) + \exp(-ipy)$ gives the decomposition

$$v_j(t,y) = v_j^+(t,y) + v_j^-(t,y) \tag{5.37}$$

where

$$v_j^+(t,y) = v_j^-(t,-y) = \frac{1}{(2\pi)^{1/2}} \int_{R_+} e^{i(yp-t\omega_j(p))} \hat{h}_j(p)\,dp \tag{5.38}$$

and the integral converges in $L_2(R_+)$ (and in $L_2(R)$) for each $h_j \in L_2(R_+)$. The special case of the fundamental mode is discussed first.

5.7 Asymptotic wave functions for the fundamental mode

This case is closely related to that of section 3, since $\omega_0(p) = p$ for all $p \in R_+$. Thus

$$v_0^+(t,y) = \frac{1}{(2\pi)^{1/2}} \int_{R_+} e^{i(y-t)p}\, \hat{h}_0(p)\,dp = G_0(y-t) \tag{5.39}$$

where

$$G_0(y) = \frac{1}{(2\pi)^{1/2}} \int_{R_+} e^{iyp}\, \hat{h}_0(p)\,dp \in L_2(R) \tag{5.40}$$

Moreover, it is easy to verify by direct calculation that $v_0^-(t,y) = v_0^+(t,-y) = G_0(-y-t) \to 0$ in $L_2(R_+)$ when $t \to \infty$. Thus

$$v_0^\infty(t,y) = G_0(y-t) \tag{5.41}$$

is an asymptotic wave function for $v_0(t,y)$ in $L_2(R_+)$:

$$\lim_{t\to\infty} \| v_0(t,\cdot) - v_0^\infty(t,\cdot)\|_{L_2(R_+)} = 0 \tag{5.42}$$

for all $\hat{h}_0 \in L_2(R_+)$.

For the higher order modes $j \geq 1$ the functions $\omega_j(p) = (p^2 + \lambda_j)^{1/2}$ with $\lambda_j > 0$. For these cases the spectral integrals (5.38) all have the same form, differing only in the value of λ_j and the function $\hat{h}_j \in L_2(R_+)$. The asymptotic behavior of these integrals may be determined by the method of stationary phase, as follows.

5.8 Application of the method of stationary phase

Consider the wave function defined by

$$v(t,y,\lambda,h) = (2\pi)^{-1/2} \int_{R_+} \exp\{i(yp - t\omega(p,\lambda))\}h(p)dp \quad (5.43)$$

where

$$\omega(p,\lambda) = (p^2 + \lambda)^{1/2} \geq \lambda^{1/2} > 0 \quad (5.44)$$

and

$$h \in L_2(R_+) \quad (5.45)$$

The phase function

$$\theta(p,\lambda,y,t) = yp - t\omega(p,\lambda) \quad (5.46)$$

is stationary with respect to p if and only if

$$\partial\theta(p,\lambda,y,t)/\partial p = y - t\partial\omega(p,\lambda)/\partial p = 0 \quad (5.47)$$

or

$$\frac{y}{t} = \frac{\partial\omega(p,\lambda)}{\partial p} \equiv U(p,\lambda) = \frac{p}{(p^2+\lambda)^{1/2}} \quad (5.48)$$

The function $U(p,\lambda)$ defined by (5.48) is the group velocity [5] for the wave function (5.43). Note that

$$\frac{\partial U(p,\lambda)}{\partial p} = \frac{\partial^2\omega(p,\lambda)}{\partial p^2} = \frac{\lambda}{(p^2+\lambda)^{3/2}} > 0 \quad (5.49)$$

and hence $U(p,\lambda)$ is a monotone increasing function of p. Moreover,

$$0 \leq U(p,\lambda) < 1 \text{ for all } p \geq 0 \text{ and } \lambda > 0 \quad (5.50)$$

Hence for $t > 0$ equation (5.48) has the unique solution

$$p = \left(\frac{(y/t)^2}{1-(y/t)^2}\lambda\right)^{1/2} = \left(\frac{y^2\lambda}{t^2 - y^2}\right)^{1/2} \geq 0 \quad (5.51)$$

if

$$0 \leq y/t < 1 \quad (5.52)$$

and has no solution for other positive values of t. The principle of stationary phase asserts that for large values of $y^2 + t^2$ the stationary point (5.51) will make a contribution

$$v^{\infty}(t,y,\lambda,h) = \chi\left(\frac{y}{t}\right) \frac{e^{i(yp-t\omega(p,\lambda)-\pi/4)}}{(t\partial U(p,\lambda)/\partial p)^{1/2}} h(p), \tag{5.53}$$

$$p = \left(\frac{y^2\lambda}{t^2 - y^2}\right)^{1/2}$$

to the integral (5.43), where $\chi(y/t)$ is the characteristic function of the set (5.52). More precisely, if $h \in \mathcal{D}(R_+)$ then the following error estimate is known [2,23].

5.9 Theorem

Let $h \in \mathcal{D}(R_+)$ and define the remainder $q(t,y,\lambda,h)$ by

$$v(t,y,\lambda,h) = v^{\infty}(t,y,\lambda,h) + q(t,y,\lambda,h) \tag{5.54}$$

Then there exists a constant $C = C(\lambda,h)$ such that

$$|q(t,y,\lambda,h)| \leq C/(y^2+t^2)^{3/4} \quad \text{for all } y \in R \text{ and } t > 0 \tag{5.55}$$

It follows from (5.54) and (5.55), by direct integration, that for all $h \in \mathcal{D}(R_+)$

$$\lim_{t\to\infty} \| v(t,\cdot,\lambda,h) - v^{\infty}(t,\cdot,\lambda,h)\|_{L_2(R_+)} = 0 \tag{5.56}$$

The stationary phase method is not applicable to (5.43) when $\lambda = 0$. However, the results for this case are described by the same equations if

$$v^{\infty}(t,y,0,h) = (2\pi)^{-1/2} \int_{R_+} \exp(iyp)\, h(p)dp \tag{5.57}$$

With this notation, (5.56) with $\lambda = 0$ is equivalent to (5.42).

The estimate (5.55) implies (5.56) for all $h \in \mathcal{D}(R_+)$. For more general $h \in L_2(R_+)$ the estimate (5.55) may not hold. Nevertheless, the following results hold.

5.10 Theorem

For all $\lambda \geq 0$ and all $h \in L_2(R_+)$

$$v^{\infty}(t,\cdot,\lambda,h) \in L_2(R_+) \text{ for all } t \neq 0 \tag{5.58}$$

$$t \to v^{\infty}(t,\cdot,\lambda,h) \in L_2(R_+) \text{ is continuous for all } t \neq 0 \tag{5.59}$$

$$\|v^{\infty}(t,\cdot,\lambda,h)\|_{L_2(R_+)} \leq \|h\|_{L_2(R_+)} \text{ for all } t \neq 0 \tag{5.60}$$

Moreover, the relation (5.56) holds for all $h \in L_2(R_+)$.

Properties (5.58) - (5.60) follow from the definitions (5.53), (5.57) by direct integration. Moreover, the validity of (5.56) for all $h \in L_2(R_+)$ follows from the special case $h \in \mathcal{D}(R_+)$, the density of $\mathcal{D}(R_+)$ in $L_2(R_+)$ and the uniform boundedness in t of $\|v(t,\cdot,\lambda,h)\|_{L_2(R_+)}$ and $\|v^{\infty}(t,\cdot,\lambda,h)\|_{L_2(R_+)}$. More detailed proofs may be found in [22,40].

5.11 Asymptotic wave functions for the higher order modes

Define the modal asymptotic wave functions by

$$v_j^{\infty}(t,y) = v^{\infty}(t,y,\lambda_j,\hat{h}_j),\ j = 0,1,2,\cdots \tag{5.61}$$

Then (5.38), (5.43) and (5.56) imply

$$\lim_{t\to\infty} \|v_j^{+}(t,\cdot) - v_j^{\infty}(t,\cdot)\|_{L_2(R_+)} = 0,\ j = 0,1,2,\cdots \tag{5.62}$$

Moreover, (5.38) for v_j^- and (5.43) imply

$$\begin{aligned} v_j^-(t,y) &= v(t,-y,\lambda_j,\hat{h}_j) \\ &= (2\pi)^{-1/2} \int_{R_+} \exp\{-i(yp + t\omega(p,\lambda_j)\}\ \hat{h}_j(p)\,dp \end{aligned} \tag{5.63}$$

The stationary phase method, applied to (5.63), implies that

$$\lim_{t\to\infty} \|v_j^-(t,\cdot)\|_{L_2(R_+)} = 0 \tag{5.64}$$

because the phase $yp + t\omega(p,\lambda_j)$ in (5.63) has no stationary points when $y \geq 0$ and $t > 0$. Combining (5.37), (5.62) and (5.64) gives

$$\lim_{t\to\infty} \|v_j(t,\cdot) - v_j^{\infty}(t,\cdot)\|_{L_2(R_+)} = 0 \tag{5.65}$$

for all $\hat{h}_j \in L_2(R_+)$ and $j = 0,1,2,\cdots$. The results and the decomposition (5.31), (5.32) imply the

5.12 Asymptotic convergence theorem

For all $h \in L_2(S)$ define

$$v^{\infty}(t,x,y) = \sum_{j=0}^{\infty} v_j^{\infty}(t,y)\, \phi_j(x),\ (x,y) \in S \tag{5.66}$$

Then

$$v(t,\cdot,\cdot) \in L_2(S) \text{ for all } t \neq 0 \tag{5.67}$$

$$t \to v(t,\cdot,\cdot) \in L_2(S) \text{ is continuous for all } t \neq 0 \tag{5.68}$$

$$\|v^{\infty}(t,\cdot,\cdot)\|_{L_2(S)} \leq \|h\|_{L_2(S)} \text{ for all } t \neq 0 \tag{5.69}$$

and

$$\lim_{t\to\infty} \|v(t,\cdot,\cdot) - v^{\infty}(t,\cdot,\cdot)\|_{L_2(S)} = 0 \tag{5.70}$$

The proof of this result will be outlined. First, note that the convergence in $L_2(S)$ of the series in (5.66) follows from the orthogonality of its terms in $L_2(S)$, (5.60) which implies

$$\|v_j^{\infty}(t,\cdot)\phi_j\|_{L_2(S)} = \|v_j^{\infty}(t,\cdot)\|_{L_2(R_+)} \leq \|\hat{h}_j\|_{L_2(R_+)} \tag{5.71}$$

for all $t \neq 0$ and (see (5.14))

$$\|h\|^2_{L_2(S)} = \sum_{j=0}^{\infty} \|\hat{h}_j\|^2_{L_2(R_+)} < \infty \tag{5.72}$$

Properties (5.68) and (5.69) follow from (5.59) and (5.60), applied to $v_j^{\infty}(t,y)$. *Finally, to* verify (5.70) note that for $j = 0,1,2,\cdots$

$$\begin{aligned}\|v_j(t,\cdot) - v_j^{\infty}(t,\cdot)\|_{L_2(R_+)} &\leq \|v_j(t,\cdot)\|_{L_2(R_+)} + \|v_j^{\infty}(t,\cdot)\|_{L_2(R_+)} \\ &\leq 2\|\hat{h}_j\|_{L_2(R_+)}\end{aligned} \tag{5.73}$$

for all $t \neq 0$. It follows that

$$\|v(t,\cdot,\cdot) - v^{\infty}(t,\cdot,\cdot)\|^2_{L_2(S)} = \sum_{j=0}^{\infty} \|v_j(t,\cdot) - v_j^{\infty}(t,\cdot)\|^2_{L_2(R_+)} \tag{5.74}$$

$$\leq \sum_{j=0}^{N} \| v_j(t,\cdot) - v_j^\infty(t,\cdot)\|^2_{L_2(R_+)} + 4 \sum_{j=N+1}^{\infty} \|\hat{h}_j\|^2_{L_2(R_+)} \quad (5.74 \text{ Cont})$$

for $N = 0,1,2,\cdots$. Fixing N and making $t \to \infty$ gives, by (5.65)

$$\overline{\lim_{t\to\infty}} \|v(t,\cdot,\cdot) - v^\infty(t,\cdot,\cdot)\|^2_{L_2(S)} \leq 4 \sum_{j=N+1}^{\infty} \|\hat{h}_j\|^2_{L_2(R_+)} \quad (5.75)$$

for $N = 0,1,2,\cdots$. Thus (5.70) follows from (5.72) and (5.75).

If $f \in L_2^1(S) = D(A^{1/2})$ and $g \in L_2(S)$ then the same method can be used to show convergence in energy:

$$\lim_{t\to\infty} E(u - u^\infty, S, t) = 0 \quad (5.76)$$

where $u^\infty(t,x,y) = \text{Re}\,\{v^\infty(t,x,y)\}$ but the details will not be recorded here.

6. SCATTERING BY OBSTACLES AND JUNCTIONS IN TUBULAR WAVEGUIDES

The analysis of section 5 is extended to compound tubular waveguides in this section. The mathematical problem is the initial-boundary value problem (4.1) - (4.3) for an unbounded domain $\Omega \subset R^3$ of the form

$$\Omega = \Omega_0 \cup S_1 \cup \cdots \cup S_m \quad (6.1)$$

where Ω_0 is a bounded domain and $S_1,\cdots,S_m$ are disjoint uniform semi-infinite cylinders. Examples include waveguides of the types described at the beginning of section 5 and many others. It will be assumed that $\Omega \in LC$.

6.1 Notation

It will be convenient to think of R^3 as a 3-dimensional differentiable manifold. The generic point of R^3 will be denoted by q. A special Cartesian coordinate system

$$(x_1^\alpha, x_2^\alpha, y^\alpha) \equiv (x^\alpha, y^\alpha) \in R^3 \quad (6.2)$$

may be associated with each semi-infinite cylinder S_α $(\alpha = 1,\cdots,m)$ in such a way that

$$S_\alpha = \{q \in R^3 \colon \ x^\alpha(q) \in G_\alpha \text{ and } y^\alpha(q) > 0\} \quad (6.3)$$

where $G_\alpha \subset R$ is a bounded domain. The assumption that $\Omega \in LC$ implies that $G_\alpha \in LC$ for $\alpha = 1,\cdots,m$ and hence that each $A(G_\alpha)$ has a discrete spectrum with eigenvalues

$$0 = \lambda_{\alpha 0} \le \lambda_{\alpha 1} \le \lambda_{\alpha 2} \le \cdots \tag{6.4}$$

such that

$$\lim_{\ell\to\infty} \lambda_{\alpha\ell} = \infty \tag{6.5}$$

and corresponding eigenfunctions

$$\phi_{\alpha 0}(x^\alpha) = 1/|G_\alpha|^{1/2},\quad \phi_{\alpha 1}(x^\alpha),\ \phi_{\alpha 2}(x^\alpha),\cdots \tag{6.6}$$

which form a complete orthonormal sequence in $L_2(G_\alpha)$.

6.2 Solutions of $Aw = \lambda w$ in S_α

Suppose that w is locally in $D(A)$; i.e., $\phi w \in D(A)$ for every $\phi \in \mathcal{D}(R^3)$. Then the completeness of the eigenfunctions (6.6) implies that

$$w(q) = \sum_{\ell=0}^{\infty} w_{\alpha\ell}(y^\alpha)\ \phi_{\alpha\ell}(x^\alpha) \text{ for all } q \in S_\alpha \tag{6.7}$$

where $x^\alpha = x^\alpha(q)$, $y^\alpha = y^\alpha(q)$. Moreover, if

$$Aw = \lambda w \text{ in } S_\alpha \tag{6.8}$$

then the coefficients $w_{\alpha\ell}(y^\alpha)$ will satisfy

$$w''_{\alpha\ell}(y^\alpha) + (\lambda - \lambda_{\alpha\ell})\ w_{\alpha\ell}(y^\alpha) = 0 \text{ for all } y^\alpha > 0 \tag{6.9}$$

In particular, if it is assumed that

$$\lambda \ne \lambda_{\alpha\ell};\ \alpha = 1,\cdots,m;\ \ell = 0,1,2,\cdots \tag{6.10}$$

then

$$w_{\alpha\ell}(y^\alpha) = C^+_{\alpha\ell} \exp\{i\sqrt{\lambda-\lambda_{\alpha\ell}}\ y^\alpha\} + C^-_{\alpha\ell} \exp\{-i\sqrt{\lambda-\lambda_{\alpha\ell}}\ y^\alpha\} \tag{6.11}$$

where, for definiteness, $\mu^{1/2} > 0$ for $\mu > 0$ and

$$\sqrt{\lambda-\lambda_{\alpha\ell}} = \begin{cases} (\lambda-\lambda_{\alpha\ell})^{1/2} & \text{for } \lambda > \lambda_{\alpha\ell} \\ i(\lambda_{\alpha\ell}-\lambda)^{1/2} & \text{for } \lambda < \lambda_{\alpha\ell} \end{cases} \tag{6.12}$$

6.3 Eigenfunctions of A and non-propagating modes

It was discovered by F. Rellich [27] that the operators A for waveguide regions of the form (6.1) may have a point spectrum. A point $\lambda \in R$ is in the point spectrum of A if and only if there is a non-zero function $w \in D(A)$ such that $Aw = \lambda w$. In particular, the requirement that $w \in L_2(\Omega)$ implies that in the expansions (6.7), (6.11) the coefficients $C^+_{\alpha\ell} = C^-_{\alpha\ell} = 0$ for $\lambda > \lambda_{\alpha\ell}$ and $C^-_{\alpha\ell} = 0$ for $\lambda < \lambda_{\alpha\ell}$. Thus any eigenfunction of A must have the form

$$w(q) = \sum_{\{\ell:\lambda<\lambda_{\alpha\ell}\}} C^+_{\alpha\ell} \exp\{-(\lambda_{\alpha\ell} - \lambda)^{1/2} y^\alpha\} \phi_{\alpha\ell}(x^\alpha) \qquad (6.13)$$

for all $q \in S_\alpha$. In particular, the eigenfunctions are exponentially damped in each cylinder S_α.

D. S. Jones [17] has shown that the point spectrum of A is a discrete subset of $(0,\infty)$; i.e., each eigenvalue has finite multiplicity and each finite subinterval of $(0,\infty)$ contains at most a finite number of eigenvalues. Thus if the point spectrum of A is not empty then there exists an M such that $1 \le M \le \infty$ and $\lambda_{(n)}$, $1 \le n < M$, is an enumeration of the eigenvalues of A, each repeated according to its multiplicity. It may be assumed that

$$0 < \lambda_{(n)} \le \lambda_{(n+1)} \text{ for } 1 \le n < n+1 < M \qquad (6.14)$$

The corresponding eigenfunctions will be denoted by $w_{(n)}$. The subspace spanned by $\{w_{(n)}: 1 \le n < M\}$ will be denoted by $\mathcal{H}^p(A)$ and called the subspace of discontinuity of A [18]. Thus

$$\mathcal{H}^p(A) = \{w = \sum_{1\le n<M} c_n w_{(n)} : \sum_{1\le n<M} |c_n|^2 < \infty\} \qquad (6.15)$$

It is known that

$$L_2(\Omega) = \mathcal{H}^p(A) \oplus \mathcal{H}^c(A) \qquad (6.16)$$

where $\mathcal{H}^c(A)$, the orthogonal complement of $\mathcal{H}^p(A)$ in $L_2(\Omega)$, is that largest subspace of $L_2(\Omega)$ on which the spectral measure of A is continuous. $\mathcal{H}^c(A)$ is called the subspace of continuity of A [18]. Moreover, (6.16) is a reducing decomposition for A [18].

If the initial state of an acoustic field in Ω satisfies $u(0,\cdot) = f \in \mathcal{H}^p(A)$ and $\partial u(0,\cdot)/\partial t = g \in \mathcal{H}^p(A)$ then $h = f + iA^{-1/2}g \in \mathcal{H}^p(A)$ and hence

$$v(t,q) = \exp(-itA^{1/2})\, h(q)$$

$$= \sum_{1 \le n < M} c_{(n)} \exp(-it\lambda_{(n)}^{1/2})\, w_{(n)}(q) \tag{6.17}$$

It follows that the energy of the acoustic field $u(t,q) = \mathrm{Re}\{v(t,q)\}$ in any bounded portion of Ω is an oscillatory function of t. In particular, there is no propagation of energy in the cylinders S_α. For this reason the eigenfunctions $w_{(n)}(q)$ are called non-propagating modes of the waveguide. By contrast, it is shown below that for fields with initial state in $\mathcal{H}^c(A)$ the energy in every bounded portion of Ω tends to zero when $t \to \infty$ and hence all the energy propagates outward in the cylinders S_α.

6.4 Generalized eigenfunctions of A

The operator A has two families of generalized eigenfunctions, analogous to the functions $w_+(x,p)$ and $w_-(x,p)$ of section 4, each of which spans the subspace $\mathcal{H}^c(A)$. The structure and properties of these functions are described next.

Consider a single term in the expansion (6.7) for the cylinder S_α. It has the form (cf. (6.11))

$$w_{\alpha\ell}(q) = (C_{\alpha\ell}^{+} \exp\{i\sqrt{\lambda - \lambda_{\alpha\ell}}\; y^\alpha\} + C_{\alpha\ell}^{-} \exp\{-i\sqrt{\lambda - \lambda_{\alpha\ell}}\; y^\alpha\})\, \phi_{\alpha\ell}(x^\alpha) \tag{6.18}$$

where $q \Leftrightarrow (x^\alpha, y^\alpha)$. Assume that $\lambda > \lambda_{\alpha\ell}$, so that (6.18) represents a propagating mode in S_α, and write

$$p = (\lambda - \lambda_{\alpha\ell})^{1/2} > 0 \tag{6.19}$$

and

$$\lambda^{1/2} \equiv \omega_{\alpha\ell}(p) = (p^2 + \lambda_{\alpha\ell})^{1/2} > \lambda_{\alpha\ell}^{1/2} \tag{6.20}$$

If one associates a time-dependence $\exp\{-i\lambda^{1/2}t\} = \exp\{-i\omega_{\alpha\ell}(p)t\}$ with (6.18), as in the spectral representation of $v(t,\cdot) = \exp(-itA^{1/2})h$, then

$$w_{\alpha\ell}(q)\exp\{-i\omega_{\alpha\ell}(p)t\} = C_{\alpha\ell}^{+}\exp\{i(py^\alpha - \omega_{\alpha\ell}(p)t)\}\phi_{\alpha\ell}(x^\alpha) + C_{\alpha\ell}^{-}\exp\{-i(py^\alpha + \omega_{\alpha\ell}(p)t)\}\phi_{\alpha\ell}(x^\alpha) \tag{6.21}$$

is the sum of an outgoing wave in S_α, with coefficient $C^+_{\alpha\ell}$, and an incoming wave with coefficient $C^-_{\alpha\ell}$. For this reason, a solution of (6.8) of the form

$$C^+_{\alpha\ell} \exp\,(ipy^\alpha)\ \phi_{\alpha\ell}(x^\alpha) \tag{6.22}$$

will be called an "outgoing" wave in S_α in mode ℓ, while a solution of the form

$$C^-_{\alpha\ell} \exp\,(-ipy^\alpha)\ \phi_{\alpha\ell}(x^\alpha) \tag{6.23}$$

will be called an "incoming" wave in S_α in mode ℓ. Note that this terminology is based on the convention that the time-dependence is $\exp\,(-i\omega_{\alpha\ell}(p)t)$, as in (6.21). If a time-dependence $\exp\,(i\omega_{\alpha\rho}(p)t)$ were used it would be necessary to interchange the terms "outgoing" and "incoming."

In the case of the uniform semi-infinite cylinder of section 5, m is equal to 1 and the generalized eigenfunctions have the form

$$\begin{aligned} w_\ell(x,y,p) &= \frac{1}{(2\pi)^{1/2}} \exp\,(ipy)\ \phi_\ell(x) \\ &\quad + \frac{1}{(2\pi)^{1/2}} \exp\,(-ipy)\ \phi_\ell(x) \end{aligned} \tag{6.24}$$

Thus they are the sum of an incoming and an outgoing wave in mode ℓ, with equal amplitudes and phases. This symmetry is due to the symmetry of the waveguide. In the general case of a compound waveguide (6.1) it is possible to prescribe the amplitudes and phases of the incoming (resp., outgoing) waves in each cylinder S_α and mode ℓ. The amplitudes and phases of the outgoing (resp., incoming) waves in each cylinder S_β and mode m are thereby determined. The most useful generalized eigenfunctions are those that have an incoming (resp., outgoing) wave of prescribed amplitude and phase in a single prescribed cylinder S_α and mode ℓ. They may be described as follows.

6.5 Definition

The mode (α,ℓ)-outgoing eigenfunction for Ω is the function $w^+_{\alpha\ell}(q,p)$ defined by the properties

$$w^+_{\alpha\ell}(\cdot,p) \text{ is locally in } D(A) \tag{6.25}$$

$$(A - \omega^2_{\alpha\ell}(p))\ w^+_{\alpha\ell}(q,p) \equiv -(\Delta + \omega^2_{\alpha\ell}(p))w^+_{\alpha\ell}(q,p) = 0 \tag{6.26}$$

for all $q \in \Omega$ and

$$w^{+}_{\alpha\ell}(q,p) = \frac{\delta_{\alpha\beta}}{(2\pi)^{1/2}} \exp(-ipy^{\alpha})\ \phi_{\alpha\ell}(x^{\alpha})$$

$$+ \sum_{m=0}^{\infty} C^{+}_{\alpha\ell,\beta m}(p) \exp\{i\sqrt{p^2 + \lambda_{\alpha\ell} - \lambda_{\beta m}}\ y^{\beta}\}\phi_{\beta m}(x^{\beta}) \tag{6.27}$$

for all $q \in S_\beta$ $(\beta = 1,2,\cdots,m)$. Similarly, the mode (α,ℓ)-incoming eigenfunction for Ω is the function $w^{-}_{\alpha\ell}(q,p)$ defined by the properties that $w^{-}_{\alpha\ell}(\cdot,p)$ is locally in $D(A)$, $(\Delta + \omega^2_{\alpha\ell}(p))w^{-}_{\alpha\ell}(q,p) = 0$ for all $q \in \Omega$ and, for $q \in S_\beta$ $(\beta = 1,2,\cdots,m)$

$$w^{-}_{\alpha\ell}(q,p) = \frac{\delta_{\alpha\beta}}{(2\pi)^{1/2}} \exp(ipy^{\alpha})\ \phi_{\alpha\ell}(x^{\alpha})$$

$$+ \sum_{m=0}^{\infty} C^{-}_{\alpha\ell,\beta m}(p) \exp\{-i\sqrt{p^2 + \lambda_{\alpha\ell} - \lambda_{\beta m}}\ y^{\beta}\}\phi_{\beta m}(x^{\beta}) \tag{6.28}$$

where $-i\sqrt{p^2 + \lambda_{\alpha\ell} - \lambda_{\beta m}} < 0$ for $\lambda_{\beta m} > \lambda_{\alpha\ell} + p^2$.

The eigenfunction $w^{+}_{\alpha\ell}(q,p)$ may be interpreted physically as the steady-state acoustic field in the waveguide Ω due to a single incoming wave (6.23) in cylinder S_α and mode ℓ, with amplitude and phase defined by $C^{-}_{\alpha\ell}(p) = 1/(2\pi)^{1/2}$, and no incoming waves in the other cylinders or in the other modes of cylinder S_α. The amplitudes and phases of the corresponding outgoing waves are defined by the coefficients $C^{+}_{\alpha\ell,\beta m}(p)$ which are determined by the incident wave and the geometry of Ω. Note that, in general, an incoming wave in mode (α,ℓ) will produce outgoing waves in all the cylinders and modes; i.e., scattering produces coupling among the cylinders and modes.

The form of the exponential which multiplies $\phi_{\beta m}(x^{\beta})$ in (6.27) is determined by the requirement (6.26). Note that the sum in (6.27) includes propagating modes with $\lambda_{\beta m} < \lambda_{\alpha\ell} + p^2$ and modes "beyond cutoff" with $\lambda_{\beta m} > \lambda_{\alpha\ell} + p^2$. The latter decrease exponentially when $y^{\beta} \to \infty$.

The eigenfunctions $w^{-}_{\alpha\ell}(q,p)$ have an interpretation analogous to that of $w^{+}_{\alpha\ell}(q,p)$, but with "outgoing" and "incoming" interchanged. It is easy to verify from the defining conditions that the two families satisfy the relation

$$w^{-}_{\alpha\ell}(q,p) = \overline{w^{+}_{\alpha\ell}(q,p)} \tag{6.29}$$

The case of the uniform semi-infinite cylinder is a very special case in which $m = 1$ (so that no index α is needed) and

$$w_\ell^+(q,p) = w_\ell^-(q,p) = w_\ell(x,y,p), \quad q \Leftrightarrow (x,y) \qquad (6.30)$$

(see (6.24)). Moreover, in this case the symmetry implies that there is no coupling between different modes:

$$c_{\ell,m}^{\pm}(p) = \delta_{\ell,m}/(2\pi)^{1/2} \qquad (6.31)$$

Existence and uniqueness theorems for the eigenfunctions $w_{\alpha\ell}^{\pm}(q,p)$ were proved in [21]. The following notation will be used to formulate them.

$$Z_{\alpha\ell}(\Omega) = \{p \in R_+ : \ p^2 + \lambda_{\alpha\ell} \in \sigma_p(A)\} \qquad (6.32)$$

where $\sigma_p(A) = \{\lambda = \lambda_{(n)} : \ 1 \le n < M\}$. Similarly,

$$Z_{\alpha\ell}(G_\beta) = \{p \in R_+ : \ p^2 + \lambda_{\alpha\ell} \in \sigma(A(G_\beta))\} \qquad (6.33)$$

where $\sigma(A(G_\beta)) = \{\lambda = \lambda_{\beta\ell} : \ \ell = 1,2,\cdots\}$. Finally

$$Z_{\alpha\ell} = Z_{\alpha\ell}(\Omega) \cup \bigcup_{\beta=1}^{m} Z_{\alpha\ell}(G_\beta) \qquad (6.34)$$

and

$$Z = \bigcup_{\alpha=1}^{m} \bigcup_{\ell=1}^{\infty} Z_{\alpha\ell} \qquad (6.35)$$

Note that the information on the spectra of $A(G_\alpha)$ and A given above implies that each of these sets is a denumerable subset of R_+. The results of [21] imply the following theorem.

6.6 Theorem

Let $\Omega \in LC$ be a waveguide domain of the form (6.1). Then for each $p \in R_+ - Z$, each $\alpha = 1,\cdots,m$ and each $\ell = 1,2,\cdots$ the eigenfunctions $w_{\alpha\ell}^+(\cdot,p)$ and $w_{\alpha\ell}^-(\cdot,p)$ exist and are unique.

6.7 The eigenfunction expansion theorem

The families $\{w_{\alpha\ell}^+(\cdot,p) : \ p \in R_+ - Z;\ \alpha = 1,\cdots,m;\ \ell = 0,1,2,\cdots\}$ and $\{w_{\alpha\ell}^-(\cdot,p) : \ p \in R_+ - Z;\ \alpha = 1,\cdots,m;\ \ell = 0,1,2,\cdots\}$ define two complete sets of generalized eigenfunctions for the part of A in the subspace of continuity $\mathcal{H}^c(A)$. The eigenfunction expansions, which are of the Plancherel type described in the preceding sections, may be formulated as follows.

6.8 Theorem

Define

$$S_{\alpha,M} = \{q \in R^3: \ x^\alpha(q) \in G_\alpha \text{ and } 0 < y^\alpha(q) < M\} \tag{6.36}$$

and

$$\Omega_M = \Omega_0 \cup S_{1,M} \cup \cdots \cup S_{m,M} \tag{6.37}$$

Then for all $f \in L_2(\Omega)$ the limits

$$\hat{f}_{\alpha\ell}(p) = L_2(R_+)\text{-}\lim_{M\to\infty} \int_{\Omega_M} \overline{w^{\pm}_{\alpha\ell}(q,p)}\ f(q)\ dV_q \tag{6.38}$$

exist, where dV_q is the element of Lebesgue measure in R^3. Moreover, the operators $\Phi^{\pm}_{\alpha\ell}$: $L_2(\Omega) \to L_2(R_+)$ defined by $\Phi^{\pm}_{\alpha\ell} f = \hat{f}^{\pm}_{\alpha\ell}$ have range $L_2(R_+)$ and, if P^c denotes the orthogonal projection of $L_2(\Omega)$ onto $\mathcal{H}^c(A)$ then

$$\|P^c f\|^2_{L_2(S)} = \sum_{\alpha=1}^{m} \sum_{\ell=0}^{\infty} \|\hat{f}^{\pm}_{\alpha\ell}\|^2_{L_2(R_+)} \tag{6.39}$$

for all $f \in L_2(\Omega)$, and

$$P^c f(q) = L_2(\Omega)\text{-}\lim_{M,N\to\infty} \sum_{\alpha=1}^{m} \sum_{\ell=0}^{N} \int_0^M w^{\pm}_{\alpha\ell}(q,p)\ \hat{f}^{\pm}_{\alpha\ell}(p)dp \tag{6.40}$$

The relations (6.38) and (6.40) will be written in the symbolic form

$$\hat{f}^{\pm}_{\alpha\ell}(p) = \int_{\Omega} \overline{w^{\pm}_{\alpha\ell}(q,p)}\ f(q)\ dV_q \tag{6.41}$$

$$P^c f(q) = \sum_{\alpha=1}^{m} \sum_{\ell=0}^{\infty} \int_{R_+} w^{\pm}_{\alpha\ell}(q,p)\ \hat{f}^{\pm}_{\alpha\ell}(p)dp \tag{6.42}$$

but they must be understood in the sense of (6.38), (6.40). The following corollaries are almost immediate; see [13,21].

6.9 Corollary

For each $f \in L_2(\Omega)$ the limits

$$f^{\pm}_{\alpha\ell}(q) = L_2(\Omega)\text{-}\lim_{M\to\infty} \int_0^M w^{\pm}_{\alpha\ell}(q,p)\ \hat{f}^{\pm}_{\alpha\ell}(p)\,dp \tag{6.43}$$

exist and

$$(f^{\pm}_{\alpha\ell}, f^{\pm}_{\beta m})_{L_2(\Omega)} = 0 \text{ whenever } (\alpha,\ell) \neq (\beta,m) \tag{6.44}$$

Moreover,

$$P^c f = \sum_{\alpha=1}^{m} \sum_{\ell=1}^{\infty} f^{\pm}_{\alpha\ell} \tag{6.45}$$

6.10 Corollary

Define

$$\mathcal{H}^{\pm}_{\alpha\ell} = \{f^{\pm}_{\alpha\ell} \in L_2(\Omega): \quad f \in L_2(\Omega)\} \tag{6.46}$$

Then each $\mathcal{H}^{\pm}_{\alpha\ell}$ is a closed subspace of $\mathcal{H}^c(A)$, $\mathcal{H}^{\pm}_{\alpha\ell}$ and $\mathcal{H}^{\pm}_{\beta m}$ are orthogonal whenever $(\alpha,\ell) \neq (\beta,m)$ and

$$\mathcal{H}^c(A) = \sum_{\alpha=1}^{m} \sum_{\ell=1}^{\infty} \oplus\, \mathcal{H}^{+}_{\alpha\ell} = \sum_{\alpha=1}^{m} \sum_{\ell=1}^{\infty} \oplus\, \mathcal{H}^{-}_{\alpha\ell} \tag{6.47}$$

The eigenfunction expansions (6.40) provide the following construction of the spectral family of A in $\mathcal{H}^c(A)$.

6.11 Theorem

If $\{\Pi(\lambda): \ \lambda \geq 0\}$ denotes the spectral family of A then

$$\Pi(\lambda)P^c f(q) = \sum_{\alpha=1}^{m} \sum_{\lambda_{\alpha\ell}\leq\lambda} \int_0^{\sqrt{\lambda-\lambda_{\alpha\ell}}} w^{\pm}_{\alpha\ell}(q,p)\ \hat{f}^{\pm}_{\alpha\ell}(p)\,dp \tag{6.48}$$

for all $\lambda \geq 0$. In particular, AP^c is an absolutely continuous operator whose spectrum is $[0,\infty)$.

Note that the sums in (6.48) are actually finite because $\lambda_{\alpha\ell} \to \infty$ when $\ell \to \infty$. (6.48) implies that the eigenfunction expansions (6.40) define spectral representations for A in the sense of the following corollary.

6.12 Corollary

If $\Psi(\lambda)$ is any bounded Lebesgue-measurable function of $\lambda \geq 0$ then for all $f \in \mathcal{H}^c(A) = P^c L_2(\Omega)$

$$\Psi(A)f(q) = \sum_{\alpha=1}^{m} \sum_{\ell=1}^{\infty} \int_{R_+} w_{\alpha\ell}^{\pm}(q,p)\, \Psi(p^2 + \lambda_{\alpha\ell})\hat{f}_{\alpha\ell}^{\pm}(p)dp \quad (6.49)$$

It follows from (6.47) and (6.49) that the eigenfunction expansions (6.40) define reducing decompositions of $\mathcal{H}^c(A)$. More precisely, the following generalization of the results of section 5 is valid [13,21].

6.13 Corollary

The operator $P_{\alpha\ell}^{\pm}$ defined by $P_{\alpha\ell}^{\pm}f = f_{\alpha\ell}^{\pm}$ is an orthogonal projections of $L_2(\Omega)$ onto $\mathcal{H}_{\alpha\ell}^{\pm}$ and

$$P^c = \sum_{\alpha=1}^{m} \sum_{\ell=1}^{\infty} P_{\alpha\ell}^{\pm} \quad (6.50)$$

Moreover,

$$P_{\alpha\ell}^{\pm}\Pi(\lambda) = \Pi(\lambda)P_{\alpha\ell}^{\pm} \text{ for all } \lambda \geq 0 \quad (6.51)$$

and hence (6.47) defines reducing decompositions for AP^c.

The surjectivity of $\Phi_{\alpha\ell}^{\pm}$: $L_2(\Omega) \to L_2(R_+)$, the definition of $P_{\alpha\ell}^{\pm}$ and (6.43) imply that for all $\alpha = 1,\cdots,m$ and $\ell = 1,2,\cdots$

$$P_{\alpha\ell}^{\pm} = \Phi_{\alpha\ell}^{\pm *}\, \Phi_{\alpha\ell}^{\pm}, \quad \Phi_{\alpha\ell}^{\pm}\, \Phi_{\alpha\ell}^{\pm *} = 1 \quad (6.52)$$

In particular the eigenfunction mappings $\Phi_{\alpha\ell}^{\pm}$ are partial isometries [18] with initial sets $\mathcal{H}_{\alpha\ell}^{\pm}$ and final sets $L_2(R_+)$.

6.14 Solution in $\mathcal{H}^c(A)$ of the propagation problem

Only the case where $f \in L_2(\Omega)$ and $g \in D(A^{-1/2})$ will be discussed here. For such initial states it follows, just as in sections 3, 4 and 5, that the solution in $L_2(\Omega)$ of (4.1) - (4.3) has the form

$$u(t,q) = \mathrm{Re}\,\{v(t,q)\} \tag{6.53}$$

where

$$v(t,\cdot) = \exp\,(-itA(\Omega)^{1/2})h, \quad h = f + iA(\Omega)^{-1/2}g \in L_2(\Omega) \tag{6.54}$$

Moreover, the case where $h \in \mathcal{H}^p(A)$ was discussed above. Hence, only the case where $h \in \mathcal{H}^c(A)$ remains to be analyzed. In this case $v(t,\cdot) \in \mathcal{H}^c(A)$ for all $t \in R$ and (6.47) implies that $v(t,q)$ has the decompositions

$$v(t,q) = \sum_{\alpha=1}^{m} \sum_{\ell=1}^{\infty} v_{\alpha\ell}^{\pm}(t,q) \text{ in } \mathcal{H}^c(A) \tag{6.55}$$

where

$$\begin{aligned} v_{\alpha\ell}^{\pm}(t,q) &= \exp\,(-itA(\Omega)^{1/2})h_{\alpha\ell}^{\pm} \\ &= \int_{R_+} w_{\alpha\ell}^{\pm}(q,p)\, \exp\,(-it\omega_{\alpha\ell}(p))\, \hat{h}_{\alpha\ell}^{\pm}(p)dp \end{aligned} \tag{6.56}$$

and $\omega_{\alpha\ell}(p)$ is given by (6.20). The two decompositions defined by (6.55), (6.56) will be called modal decompositions, in analogy with the simple case of section 5, and the partial wave $v_{\alpha\ell}^{\pm}(t,q)$ will be said to be in mode $(\pm,\alpha,\ell)$ of the compound waveguide Ω. Note that for the uniform semi-infinite cylinder of section 5 the $(+,\ell)$ and $(-,\ell)$ modes coincide (see (6.30)).

6.15 Transiency of waves in $\mathcal{H}^c(A)$

The absolute continuity of the operator A in the subspace $\mathcal{H}^c(A)$ implies that all waves in $\mathcal{H}^c(A)$ are transient in the sense of the following theorem [38,42].

6.16 Theorem

If $\Omega \in LC$ is a waveguide domain (6.1) then for every $h \in \mathcal{H}^c(A)$ and every compact set $K \subset R^3$

$$\lim_{t\to\infty} \| \exp(-itA^{1/2})h \|_{L_2(K\cap\Omega)} = 0 \tag{6.57}$$

Thus the decomposition $L_2(\Omega) = \mathcal{H}^p(A) \oplus \mathcal{H}^c(A)$ splits every $h \in L_2(\Omega)$ into a sum of a non-propagating and a propagating state. In particular the partial waves

$$v^{\pm}_{\alpha\ell}(t,\cdot) = \exp(-itA^{1/2})\, P^{\pm}_{\alpha\ell} h \in \mathcal{H}^c(A) \tag{6.58}$$

and hence (6.57) with $K = \overline{\Omega_0}$ implies

$$\lim_{t\to\infty} \| v^{\pm}_{\alpha\ell}(t,\cdot) \|_{L_2(\Omega_0)} = 0 \tag{6.59}$$

for $\alpha = 1,2,\cdots,m$ and $\ell = 0,1,2,\cdots$. Thus waves in $\mathcal{H}^c(A)$ ultimately propagate into the cylinders S_α. The eigenfunction expansion for A will now be used to calculate the asymptotic form of these waves.

6.17 Asymptotic wave functions

Let $h \in \mathcal{H}^c(A)$ and consider the representation

$$v(t,\cdot) = \exp(-itA^{1/2})h = \sum_{\alpha=1}^{m} \sum_{\ell=0}^{\infty} v^{-}_{\alpha\ell}(t,\cdot) \tag{6.60}$$

defined by the incoming eigenfunctions $w^{-}_{\alpha\ell}(q,p)$. Substituting the development (6.28) for $w^{-}_{\alpha\ell}(q,p)$ in S_β into the integral (6.56) for $v^{-}_{\alpha\ell}(t,q)$ gives the representation

$$v^{-}_{\alpha\ell}(t,q) = \delta_{\alpha\beta} v(t,y^\alpha,\lambda_{\alpha\ell},\hat{h}^{-}_{\alpha\ell})\phi_{\alpha\ell}(x^\alpha) + v'_{\alpha\ell}(t,q), \quad q \in S_\beta \tag{6.61}$$

where $v(t,y,\lambda,h)$ is defined by (5.43) and

$$v'_{\alpha\ell}(t,q) = \sum_{m=0}^{\infty} v'_{\alpha\ell,\beta m}(t,y^\beta)\, \phi_{\beta m}(x^\beta), \quad q \in S_\beta \tag{6.62}$$

with

$$v'_{\alpha\ell,\beta m}(t,y) = \int_0^{\sqrt{\lambda_{\beta m}-\lambda_{\alpha\ell}}} \exp\left(-\sqrt{\lambda_{\beta m}-\lambda_{\alpha\ell}-p^2}\,y\right) \times \exp\left(-it\omega_{\alpha\ell}(p)\right) C^-_{\alpha\ell,\beta m}(p)\hat{h}^-_{\alpha\ell}(p)dp$$

$$+ \int_{\sqrt{\lambda_{\beta m}-\lambda_{\alpha\ell}}}^{\infty} \exp\left\{-i\left(\sqrt{p^2+\lambda_{\alpha\ell}-\lambda_{\beta m}}\,y + t\omega_{\alpha\ell}(p)\right)\right\} \times C^-_{\alpha\ell,\beta m}(p)\,\hat{h}^-_{\alpha\ell}(p)dp \tag{6.63}$$

This equation defines $v'_{\alpha\ell,\beta m}$ for the case where $\lambda_{\beta m} > \lambda_{\alpha\ell}$. In the case where $\lambda_{\beta m} \le \lambda_{\alpha\ell}$ the first integral is absent and the second has the lower limit zero. The method of stationary phase may be applied to show that $v'_{\alpha\ell,\beta m}(t,y)$ satisfies an estimate of the form (5.55) for $y \ge 0$, $t \ge 0$ because the integrals in (6.63) have no points of stationary phase in this region. It follows that

$$\lim_{t\to\infty} \| v'_{\alpha\ell}(t,\cdot)\|_{L_2(S_\beta)} = 0, \quad \beta = 1,2,\cdots,m \tag{6.64}$$

The proof may be based on a convergence lemma like that of section 3. The details will not be given here. See [22] for a more complete discussion.

In (6.61) the term $v(t,y^\alpha,\lambda_{\alpha\ell},\hat{h}^-_{\alpha\ell})$ has the form (5.43) studied in section 5. Hence, if $v^\infty(t,y,\lambda,h)$ is defined as in that section and

$$v^\infty_{\alpha\ell}(t,q) = v^\infty(t,y^\alpha,\lambda_{\alpha\ell},\hat{h}^-_{\alpha\ell})\ \phi_{\alpha\ell}(x^\alpha), \quad q \in S_\alpha \tag{6.65}$$

then (6.64) and the results of section 5 imply

$$\lim_{t\to\infty} \| v^-_{\alpha\ell}(t,\cdot) - \delta_{\alpha\beta} v^\infty_{\alpha\ell}(t,\cdot)\|_{L_2(S_\beta)} = 0, \quad \beta = 1,2,\cdots,m \tag{6.66}$$

In particular, $v^-_{\alpha\ell}(t,\cdot) \to 0$ in $L_2(S_\beta)$ for $t \to \infty$ and all $\beta \ne \alpha$; i.e., $v^-_{\alpha\ell}(t,\cdot)$ is asymptotically concentrated entirely in S_α.

The asymptotic wave functions for $v(t,q)$ will be defined by

$$v^{\infty}(t,q) = \sum_{\alpha=1}^{m} \sum_{\ell=0}^{\infty} \chi_{\alpha}(q)\, v^{\infty}_{\alpha\ell}(t,q), \quad q \in \Omega \tag{6.67}$$

where $\chi_\alpha(q)$ denotes the characteristic function of S_α. Note that the terms in this sum are orthogonal in $L_2(\Omega)$ by (6.65). The decompositions (6.60), (6.67) and the convergence results (6.59) and (6.66) imply the

6.18 Theorem

If $\Omega \in LC$ is a waveguide domain (6.1) then for every $h \in \mathcal{H}^c(A)$ the wave function $v(t,\cdot) = \exp(-itA^{1/2})h$ satisfies

$$\lim_{t\to\infty} \| v(t, \;) - v^{\infty}(t,\cdot)\|_{L_2(\Omega)} = 0 \tag{6.68}$$

The proof is essentially the same as for the special case described in section 5. The convergence in energy, when h has finite energy, can be proved by the same methods.

7. PROPAGATION IN PLANE STRATIFIED FLUIDS

The propagation of localized acoustic waves in a plane stratified fluid which fills a half-space is analyzed in this section. The asymptotic wave functions for such media are shown to be the sum of an asymptotic free (hemispherical) wave and an asymptotic guided wave which propagates parallel to the boundary. This structure, which is intermediate between that of a homogeneous fluid and that of a tubular waveguide, is called an open waveguide in the physical literature.

7.1 Plane stratified fluids

An inhomogeneous fluid will be said to be plane stratified if the local sound speed $c(x)$ and density $\rho(x)$ are functions of a single Cartesian coordinate. This condition can be written

$$\left.\begin{aligned} c(x_1,x_2,x_3) &= c(x_3) \\ \rho(x_1,x_2,x_3) &= \rho(x_3) \end{aligned}\right\} \tag{7.1}$$

with a suitable numbering of the coordinates. It will be convenient to denote x_3 by a single letter and write

$$x = (x_1, x_2) \in R^2, \; y = x_3 \in R, \; (x,y) \in R^3 \tag{7.2}$$

This notation is used in the remainder of this section.

7.2 Propagation in a stratified fluid with a free surface

A stratified fluid filling a half-space

$$R^3_+ = \{(x,y): \;\; x \in R^2 \text{ and } y > 0\} \tag{7.3}$$

is often used as a model in the study of acoustic wave propagation in oceans and deep lakes. If the surface $\{(x,0): \;\; x \in R^2\}$ is free the corresponding initial-boundary value problem is (see section 2)

$$\frac{\partial^2 u}{\partial t^2} - c^2(y)\rho(y) \frac{\partial}{\partial x_j} \left(\frac{1}{\rho(y)} \frac{\partial u}{\partial x_j}\right) = 0 \text{ for } t > 0, \quad (x,y) \in R^3_+ \tag{7.4}$$

$$u(t,x,0) = 0 \text{ for } t \geq 0, \; x \in R^2 \tag{7.5}$$

$$u(0,x,y) = f(x,y) \text{ and } \partial u(0,x,y)/\partial t = g(x,y) \quad \text{for } (x,y) \in R^3_+ \tag{7.6}$$

where in (7.4) j is summed from 1 to 3 and $x_3 = y$. $c(y)$ and $\rho(y)$ are assumed to be Lebesgue measurable on $R_+ = \{y: \;\; y > 0\}$ and to satisfy

$$\left.\begin{array}{l} 0 < c_1 \leq c(y) \leq c_2 < \infty \\ \\ 0 < \rho_1 \leq \rho(y) \leq \rho_2 < \infty \end{array}\right\} \text{ for all } y \in R_+ \tag{7.7}$$

where c_1, c_2, ρ_1 and ρ_2 are suitable constants.

7.3 Hilbert space formulation

The operator

$$Au = -c^2(y)\rho(y) \frac{\partial}{\partial x_j} \left(\frac{1}{\rho(y)} \frac{\partial u}{\partial x_j}\right) \tag{7.8}$$

was shown in section 2 to be formally selfadjoint with respect to the inner product

$$(u,v) = \int_{R_+^3} \overline{u(x,y)}\, v(x,y)\, c^{-2}(y)\, \rho^{-1}(y)dxdy \tag{7.9}$$

where dxdy denotes integration with respect to Lebesgue measure on R_+^3. The corresponding Hilbert space is

$$\mathcal{H} = L_2(R_+^3, c^{-2}(y)\, \rho^{-1}(y)dxdy) \tag{7.10}$$

The solution of the initial-boundary value problem (7.4) - (7.6) given below is based on a selfadjoint realization of A in $\mathcal{H}$. To define it let $\mathcal{D}(R_+^3)$ denote the Schwartz space of R_+^3 and $\mathcal{D}'(R_+^3)$ the dual space of all distributions on R_+^3. The Lebesgue space $L_2(R_+^3)$ can be regarded as a linear subspace of $\mathcal{D}'(R_+^3)$. Note that $L_2(R_+^3)$ and $\mathcal{H}$ are equivalent Hilbert spaces by (7.7). Let

$$L_2^1(R_+^3) = L_2(R_+^3) \cap \{u: \ \partial u/\partial x_j \in L_2(R_+^3),\ j = 1,2,3\} \tag{7.11}$$

denote the first Sobolev space of R_+^3. It is a Hilbert space with inner product

$$(u,v)_1 = (u,v)_0 + \sum_{j=1}^{3} (\partial u/\partial x_j, \partial v/\partial x_j)_0 \tag{7.12}$$

where $(u,v)_0$ is the inner product in $L_2(R_+^3)$. $\mathcal{D}(R_+^3)$ defines a linear subset of $L_2^1(R_+^3)$ and hence

$$L_2^{1,0}(R_+^3) = \text{closure of } \mathcal{D}(R_+^3) \text{ in } L_2^1(R_+^3) \tag{7.13}$$

is a closed linear subspace of $L_2^1(R_+^3)$. It is known that all the functions in $L_2^{1,0}(R_+^3)$ satisfy the Dirichlet boundary condition (7.5) as elements of $L_2(R^2)$; see [19] and [43,Cor. 2.7].

A realization of the operator A in $\mathcal{H}$ will be defined by

$$D(A) = L_2^{1,0}(R_+^3) \cap \{u: \ Au \in \mathcal{H}\} \tag{7.14}$$

and

$$Au = Au \text{ for all } u \in D(A) \tag{7.15}$$

To interpret the condition $Au \in \mathcal{H}$ in (7.14) note that if $u \in L_2^1(R_+^3)$ then $\rho^{-1}(y)\partial u/\partial x_j \in L_2(R_+^3)$ for $j = 1,2,3$. The second derivative in (7.8) may therefore be interpreted in the sense of $\mathcal{D}'(R_+^3)$. Thus $Au \in \mathcal{D}'(R_+^3)$ and the condition $Au \in \mathcal{H}$ is meaningful.

The selfadjointness of A in $\mathcal{H}$ may be proved by the method of [43,§2]. Another proof may be based on the theory of sesquilinear forms in Hilbert space [18,Ch. 6]. These methods imply the

7.4 Theorem

A is a selfadjoint real positive operator in $\mathcal{H}$. Every $u \in D(A)$ satisfies the Dirichlet boundary condition (7.5) as an element of $L_2(R^2)$. Moreover, $D(A^{1/2}) = L_2^{1,0}(R_+^3)$ and

$$\|A^{1/2}u\|^2 = \sum_{j=1}^{3} \int_{R_+^3} |\partial u/\partial x_j|^2 \rho^{-1}(y)\,dx\,dy \quad \text{for all } u \in D(A^{1/2}) \tag{7.16}$$

The operator A may be used to construct "solutions in $\mathcal{H}$" and "solutions with finite energy" of (7.4) - (7.6), as described in section 2. The detailed analysis of the structure of these solutions will again depend on the construction of an eigenfunction expansion for A. For simplicity, the construction will be described here for a special choice of the functions $c(y)$ and $\rho(y)$. Nevertheless, the results obtained are typical of a large class of stratified fluids.

7.5 The Pekeris model

This name will be used for the stratified fluid defined by

$$c(y) = \begin{cases} c_1, & 0 \le y < h \\ c_2, & y \ge h \end{cases} \tag{7.17}$$

$$\rho(y) = \begin{cases} \rho_1, & 0 \le y < h \\ \rho_2, & y \ge h \end{cases} \tag{7.18}$$

where c_1, c_2, ρ_1, ρ_2 and h are positive constants. This model was used by C. L. Pekeris [26] in his study of acoustic wave propagation in shallow water. The model represents a layer of water with depth h, sound speed c_1 and density ρ_1 which overlays a bottom, such as sand or mud, with sound speed c_2 and density ρ_2.

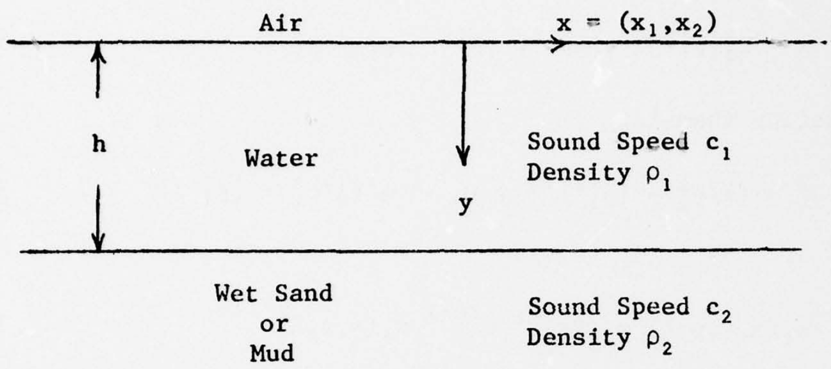

Figure 7. The Pekeris Model

The most interesting case occurs when

$$c_1 < c_2 \tag{7.19}$$

and this condition is assumed to be satisfied in what follows. A detailed study of the Pekeris operator was given by the author in [43]. Here the main results of [43] are reviewed and used to derive the asymptotic wave functions for the Pekeris model.

7.6 Eigenfunctions of A

It was shown in [43] that A has a pure continuous spectrum and a complete family of generalized eigenfunctions was constructed. These functions w(x,y) are characterized by the following properties

$$w \text{ is locally in } D(A) \tag{7.20}$$

$$Aw = \lambda w \text{ for some } \lambda \geq 0 \tag{7.21}$$

$$w(x,y) \text{ is bounded in } R^3_+ \tag{7.22}$$

$$w(x,y) = (2\pi)^{-1} e^{ip\cdot x} w(y),\ p \in R^2 \tag{7.23}$$

where, in (7.23), w(y) is independent of x. The eigenfunctions are of two types, called free wave eigenfunctions and guided wave eigenfunctions. Their definitions and physical interpretations follow.

7.7 Free wave eigenfunctions

These functions exist when the eigenvalue λ satisfies

$$\lambda > c_2^2|p|^2 > c_1^2|p|^2, \quad |p|^2 = p_1^2 + p_2^2 \tag{7.24}$$

To define them let

$$\xi = (\lambda/c_2^2 - |p|^2)^{1/2} > 0, \quad \eta = (\lambda/c_1^2 - |p|^2)^{1/2} > 0 \tag{7.25}$$

and

$$w_0(x,y,p,\lambda) = (2\pi)^{-1} e^{ip\cdot x} w_0(y,p,\lambda) \tag{7.26}$$

where

$$w_0(y,p,\lambda) = a(p,\lambda) \begin{cases} \sin \eta y & , 0 < y < h \\ \gamma_+(\xi,\eta)\, e^{i\xi(y-h)} + \gamma_-(\xi,\eta) e^{-i\xi(y-h)}, & y > h \end{cases} \tag{7.27}$$

with

$$\gamma_\pm(\xi,\eta) = \frac{1}{2}\left\{\sin \eta h \mp \frac{\rho_2}{\rho_1}\frac{i\eta}{\xi}\cos \eta h\right\} \tag{7.28}$$

In (7.27), $a(p,\lambda)$ is a positive normalizing constant. It was shown in [43] that the eigenfunction expansion takes its simplest form when

$$a(p,\lambda) = \rho_2^{1/2}/2(\pi\xi)^{1/2}|\gamma_+(\xi,\eta)| \tag{7.29}$$

In physical terms, the eigenfunction $w_0(x,y,p,\lambda)$ represents an acoustic field with time dependence $\exp(-it\lambda^{1/2})$ which is the sum of two plane waves in each layer. It may be interpreted as a plane wave which propagates in the region $y > h$, is refracted at $y = h$, reflected at $y = 0$ and refracted again at $y = h$; see Figure 8 where the propagation directions are indicated. It can be verified that Snell's law of refraction is satisfied at $y = h$ and the law of reflection is satisfied at $y = 0$.

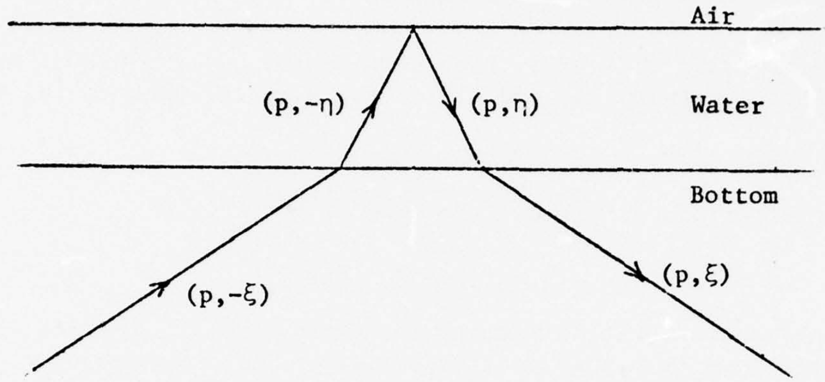

Figure 8. Ray diagram for free wave eigenfunction

7.8 The dispersion relation

For values of λ which satisfy

$$c_1^2|p|^2 < \lambda < c_2^2|p|^2 \qquad (7.30)$$

the function $w_0(x,y,p,\lambda)$ defined by (7.25) - (7.28) still satisfies conditions (7.20), (7.21) and (7.23). However, (7.30) implies that ξ is pure imaginary, say

$$\xi = i\xi', \ \xi' = (|p|^2 - \lambda/c_2^2)^{1/2} > 0 \qquad (7.31)$$

while η is still real and positive. It follows that $w_0(x,y,p,\lambda)$ satisfies the boundedness condition (7.22) if and only if

$$\gamma_-(i\xi',\eta) = 0 \qquad (7.32)$$

or, by (7.28),

$$\xi' = -\frac{\rho_2}{\rho_1}\eta \operatorname{ctn} \eta h \qquad (7.33)$$

For λ and $|p|$ which satisfy (7.30), (7.33) is equivalent to the sequence of equations

$$h\eta = (k - \frac{1}{2})\pi + \tan^{-1}\left(\frac{\rho_1\xi'}{\rho_2\eta}\right), \quad k = 1,2,\cdots \tag{7.34}$$

where $|\tan^{-1}\alpha| < \pi/2$. Each equation (7.34) defines a functional relation between $|p|$ and λ or, equivalently, between $|p|$ and

$$\omega = \lambda^{1/2} \tag{7.35}$$

The solutions, which will be denoted by

$$\lambda = \lambda_k(|p|), \quad \omega = \omega_k(|p|) = \lambda_k(|p|)^{1/2} \tag{7.36}$$

represent a relation between the wave number $|p|$ of the plane waves in $w_0(x,y,p,\lambda)$ and the corresponding frequencies ω. Such relations are called dispersion relations in the theory of wave motion. The relations (7.34), (7.36) were analyzed in [43] and found to have the following properties.

7.9 Properties of $\omega_k(|p|)$

For each $k = 1,2,3,\cdots$ define

$$p_1 = \pi c_1/2h(c_2^2 - c_1^2)^{1/2}, \quad p_k = (2k-1)p_1 \tag{7.37}$$

Then

$$\omega_k(|p|) \text{ is analytic and } \omega_k'(|p|) > 0 \text{ for } |p| \geq p_k \tag{7.38}$$

$$c_1|p| \leq \omega_k(|p|) \leq c_2|p| \text{ for } |p| \geq p_k \tag{7.39}$$

$$\omega_k(p_k) = c_2 p_k, \quad \omega_k'(p_k) = c_2 \tag{7.40}$$

$$\omega_k(|p|) \sim c_1|p| \text{ for } |p| \to \infty \tag{7.41}$$

Moreover, an explicit parametric representation of the dispersion curves (7.36) was given in [43].

7.10 Guided wave eigenfunctions

The functions $w_k(x,y,p) = w_0(x,y,p,\lambda_k(|p|))$ are, by construction, the solutions of (7.20) - (7.23) for eigenvalues which satisfy (7.30). It was shown in [43] that there are no solutions of (7.20) - (7.23) when $\lambda < c_1^2|p|^2$. The functions $w_k(x,y,p)$ have the form

$$w_k(x,y,p) = (2\pi)^{-1} e^{ip\cdot x} w_k(y,p) \tag{7.42}$$

where

$$w_k(y,p) = a_k(p) \begin{cases} \sin \eta_k(|p|)y & , \ 0 < y < h \\ \sin \eta_k(|p|)h \ e^{-\xi_k'(|p|)(y-h)} & , \ y > h \end{cases} \tag{7.43}$$

with

$$\left.\begin{aligned} \eta_k(|p|) &= (\lambda_k(|p|)/c_1^2 - |p|^2)^{1/2}, \\ \xi_k'(|p|) &= (|p|^2 - \lambda_k(|p|)/c_2^2)^{1/2} \end{aligned}\right\} \tag{7.44}$$

In (7.43), $a_k(p)$ is a positive constant which is determined by the condition

$$\int_0^\infty |w_k(y,p)|^2 \ c^{-2}(y) \ \rho^{-1}(y)dy = 1 \tag{7.45}$$

In physical terms, the eigenfunction $w_k(x,y,p)$ represents an acoustic field with time dependence exp $(-it\omega_k(|p|))$ which corresponds to a plane wave which is trapped in the layer $0 \le y \le h$ by reflection at $y = 0$ and total internal reflection at the interface $y = h$. In the layer $y > h$ the field is exponentially damped in the y-direction and propagates strictly in the horizontal direction p; see Figure 9 where the propagation directions are indicated.

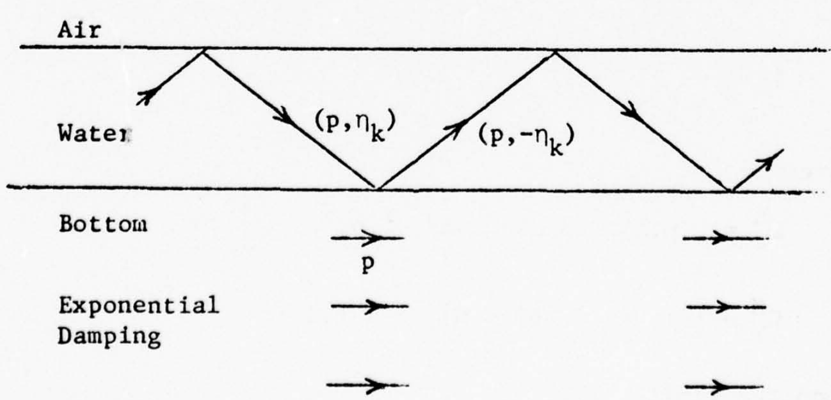

Figure 9. Ray diagram for guided wave eigenfunctions

7.11 The eigenfunction expansion

The free wave eigenfunctions $w_0(x,y,p,\lambda)$ are parameterized by the region

$$\Omega_0 = \{(p,\lambda): \ p \in R^2 \text{ and } c_2^2|p|^2 < \lambda\} \subset R^3 \tag{7.46}$$

Similarly, the guided wave eigenfunctions $w_k(x,y,p)$ are parameterized by the regions

$$\Omega_k = \{p: \ |p| > p_k\}, \ k = 1,2,\cdots \tag{7.47}$$

The following expansion theorem was proved in [43]. First of all, for every $f \in \mathcal{H}$ the limits

$$\hat{f}_0(p,\lambda) = L_2(\Omega_0)\text{-}\lim_{M\to\infty} \int_0^M \int_{|x|\le M} \overline{w_0(x,y,p,\lambda)}\, f(x,y)c^{-2}(y)\rho^{-1}(y)\,dxdy \tag{7.48}$$

and

$$\hat{f}_k(p) = L_2(\Omega_k)\text{-}\lim_{M\to\infty} \int_0^M \int_{|x|\le M} \overline{w_k(x,y,p)}\, f(x,y)c^{-2}(y)\rho^{-1}(y)\,dxdy, \quad k = 1,2,\cdots \tag{7.49}$$

exist and satisfy the Parseval relation

$$\|f\|_{\mathcal{H}}^2 = \sum_{k=0}^{\infty} \|\hat{f}_k\|_{L_2(\Omega_k)}^2 \tag{7.50}$$

Moreover, if

$$\left.\begin{aligned} \Omega_0^M &= \{(p,\lambda): \ p \in R^2 \text{ and } c_2^2|p|^2 < \lambda < M\} \\ \Omega_k^M &= \{p: \ p_k < |p| < M\}, \ k = 1,2,\cdots \end{aligned}\right\} \tag{7.51}$$

then the limits

$$f_0(x,y) = \mathcal{H}\text{-}\lim_{M\to\infty} \int_{\Omega_0^M} w_0(x,y,p,\lambda)\, \hat{f}_0(p,\lambda)dpd\lambda \tag{7.52}$$

and

$$f_k(x,y) = \mathcal{H}\text{-}\lim_{M\to\infty} \int_{\Omega_k^M} w_k(x,y,p)\, \hat{f}_k(p)dp,\ k = 1,2,\cdots \tag{7.53}$$

exists and satisfy

$$f(x,y) = \mathcal{H}\text{-}\lim_{M\to\infty} \sum_{k=0}^{M} f_k(x,y) \tag{7.54}$$

The relations (7.48), (7.49), (7.52), (7.53) and (7.54) will also be written in the following more concise symbolic forms, in analogy with the notation of previous sections.

$$\hat{f}_0(p,\lambda) = \int_{R_+^3} \overline{w_0(x,y,p,\lambda)}\, f(x,y)\, c^{-2}(y)\, \rho^{-1}(y)dxdy \tag{7.55}$$

$$\hat{f}_k(p) = \int_{R_+^3} \overline{w_k(x,y,p)}\, f(x,y)\, c^{-2}(y)\, \rho^{-1}(y)dxdy, \quad k = 1,2,\cdots \tag{7.56}$$

$$f_0(x,y) = \int_{\Omega_0} w_0(x,y,p,\lambda)\, \hat{f}_0(p,\lambda)dpd\lambda \tag{7.57}$$

$$f_k(x,y) = \int_{\Omega_k} w_k(x,y,p)\, \hat{f}_k(p)dp,\ k = 1,2,\cdots \tag{7.58}$$

$$f(x,y) = \sum_{k=0}^{\infty} f_k(x,y) \tag{7.59}$$

Equations (7.55) - (7.59) are the eigenfunction expansion for A and show the completeness of the generalized eigenfunctions defined above. The representation is a spectral representation for A in the sense that, for every $f \in D(A)$,

$$(Af)\hat{\ }_0(p,\lambda) = \lambda\, \hat{f}_0(p,\lambda) \tag{7.60}$$

and

$$(Af)\hat{}_k(p) = \lambda_k(|p|)\ \hat{f}_k(p), \quad k = 1,2,\cdots \tag{7.61}$$

The representation (7.55) - (7.59) defines a modal decomposition for the Pekeris model. It was shown in [43] that if

$$\mathcal{H}_k = \{f_k:\ f \in \mathcal{H}\} \subset \mathcal{H}, \quad k = 0,1,2,\cdots \tag{7.62}$$

then each $\mathcal{H}_k$ is a closed subspace, $\mathcal{H}_k$ and $\mathcal{H}_\ell$ are orthogonal for $k \neq \ell$ and

$$\mathcal{H} = \sum_{k=0}^{\infty} \oplus\ \mathcal{H}_k \tag{7.63}$$

Moreover, it was shown that (7.60), (7.62) imply that (7.63) reduces A. In fact, more was shown in [43]; namely that

$$\Phi_k f = \hat{f}_k \in L_2(\Omega_k), \quad k = 0,1,2,\cdots \tag{7.64}$$

defines an operator

$$\Phi_k:\ \mathcal{H} \to L_2(\Omega_k), \quad k = 0,1,2,\cdots \tag{7.65}$$

which is a partial isometry with initial set $\mathcal{H}_k$ and final set $L_2(\Omega_k)$; i.e.,

$$\Phi_k^*\ \Phi_k = P_k,\ \Phi_k\ \Phi_k^* = 1, \quad k = 0,1,2,\cdots \tag{7.66}$$

where P_k is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_k$.

7.12 Solution in $\mathcal{H}$ of the propagation problem

Attention will again be restricted to the case where $f \in \mathcal{H}$ and $g \in D(A^{-1/2})$ so that the solution in $\mathcal{H}$ has the form

$$u(t,x,y) = \mathrm{Re}\ \{v(t,x,y)\} \tag{7.67}$$

with

$$v(t,\cdot,\cdot) = \exp\ (-itA^{1/2})h, \quad h = f + iA^{-1/2}g \in \mathcal{H} \tag{7.68}$$

The modal decomposition of v(t,x,y) is

$$v(t,x,y) = \sum_{k=0}^{\infty} v_k(t,x,y) \tag{7.69}$$

where

$$v_0(t,x,y) = \int_{\Omega_0} w_0(x,y,p,\lambda) \exp(-it\lambda^{1/2})\, \hat{h}_0(p,\lambda)\,dp\,d\lambda \tag{7.70}$$

and

$$v_k(t,x,y) = \int_{\Omega_k} w_k(x,y,p) \exp(-it\omega_k(|p|))\hat{h}_k(p)\,dp, \quad k = 1,2,\cdots \tag{7.71}$$

Moreover, the modal waves $v_k(t,x,y)$ are independent in the sense that they are orthogonal in $\mathcal{H}$ for every $t \in R$ because (7.63) is a reducing decomposition of A. Asymptotic wave functions for each mode will now be calculated beginning with the guided modes v_k, $k \geq 1$.

7.13 Asymptotic wave functions for the guided modes ($k \geq 1$)

If the representation (7.42) for the eigenfunctions $w_k(x,y,p)$ is substituted into (7.71) the spectral integrals takes the form

$$v_k(t,x,y) = \frac{1}{2\pi} \int_{\Omega_k} \exp\{i(x\cdot p - t\omega_k(|p|))\} w_k(y,p)\hat{h}_k(p)\,dp, \quad k = 1,2,\cdots \tag{7.72}$$

where $w_k(y,p)$ is defined by (7.43). The behavior for large t of these integrals will be calculated by the method of stationary phase. In the present case the integral is a double integral ($\Omega_k \subset R^2$) and the phase function

$$\theta_k(p,x,t) = x \cdot p - t\omega_k(|p|) \tag{7.73}$$

is stationary with respect to p if and only if

$$\frac{\partial\theta_k(p,x,t)}{\partial p_j} = x_j - t\omega_k'(|p|)\,\frac{p_j}{|p|} = 0,\ j = 1,2 \tag{7.74}$$

In particular, the number and distribution of the stationary points is determined by the group speed function for the kth guided mode:

$$U_k(|p|) = \omega_k'(|p|),\ |p| \geq p_k \tag{7.75}$$

The defining relation (7.34) for $\omega_k(|p|)$ implies the following

7.14 Properties of $U_k(|p|)$

For each $k = 1,2,3,\cdots$ there exists a unique $p_k^A \geq p_k$ where $U_k'(p_k^A) = 0$ and $p_k^A > p_k$. Moreover,

$$0 < U_k^A \equiv U_k(p_k^A) \leq U_k(|p|) \leq c_2 \text{ for all } |p| \geq p_k \tag{7.76}$$

$$U_k'(|p|) < 0 \text{ for } p_k \leq |p| < p_k^A \text{ and } U_k'(|p|) > 0 \text{ for } |p| > p_k^A \tag{7.77}$$

$$\lim_{|p|\to p_k} U_k(|p|) = c_2,\quad \lim_{|p|\to\infty} U_k(|p|) = c_1 \tag{7.78}$$

These properties are indicated in Figure 10.

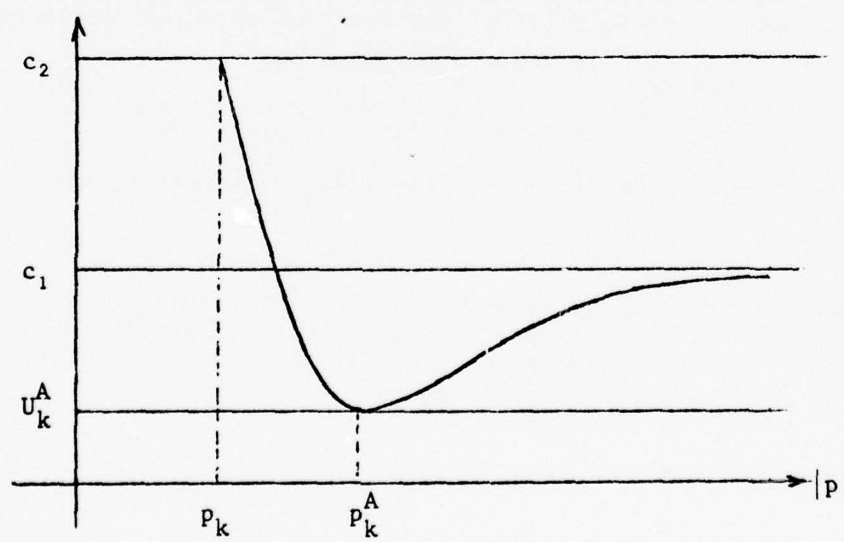

Figure 10. The Group Speed Curve $q = U_k(|p|)$

The stationary points of $\theta_k(p,x,t)$ are defined by (7.74). This may be written in (2-dimensional) vector notation

$$x/t = U_k(|p|)p/|p| \tag{7.79}$$

This is equivalent to the conditions

$$U_k(|p|) = |x|/t \tag{7.80}$$

and

$$p \text{ is parallel to } x \text{ and in the same direction} \tag{7.81}$$

since $U_k(|p|) > 0$ and t is assumed to be positive. Conditions (7.80) and (7.81) determine $|p|$ and $p/|p|$, respectively. In particular, it is clear from Figure 10 that

$$\left.\begin{array}{l} \text{For } |x| > c_2 t \text{ and } |x| < U_k^A t \text{ there are no points of} \\ \quad \text{stationary phase} \\[1ex] \text{For } c_1 t \leq |x| \leq c_2 t \text{ and } |x| = U_k^A t \text{ there is one point} \\ \quad \text{of stationary phase} \\[1ex] \text{For } U_k^A t < |x| < c_1 t \text{ there are two points of} \\ \quad \text{stationary phase} \end{array}\right\} \tag{7.82}$$

According to the method of stationary phase each stationary point where $\det(\partial^2\theta_k/\partial p_i \partial p_j) \neq 0$ (regular stationary point) contributes a term

$$\frac{\exp\left(i\theta_k(p,x,t) + i\,\frac{\pi}{4}\,\mathrm{sgn}\,(\partial^2\theta_k/\partial p_i\partial p_j)\right)}{|\det\,(\partial^2\theta_k/\partial p_i\partial p_j)|^{1/2}}\; w_k(y,p)\hat{h}_k(p) \tag{7.83}$$

to the asymptotic expansion of the integral (7.72), where sgn and det denote the signature and determinant, respectively, of the Gramian matrix $(\partial^2\theta_k/\partial p_i\partial p_j)$. A short calculation shows that the eigenvalues of the Gramian for (7.73) are $-tU_k'(|p|)$ and $-tU_k(|p|)/|p|$ and hence for $t > 0$

$$\mathrm{sgn}\,(\partial^2\theta_k/\partial p_i\partial p_j) = -1 - \mathrm{sgn}\,U_k'(|p|) \tag{7.84}$$

$$\det\,(\partial^2\theta_k/\partial p_i\partial p_j) = t^2 U_k(|p|)U_k'(|p|)/|p| \tag{7.85}$$

In particular, the stationary points are regular when $|p| \neq p_k^A$. Substitution of (7.73), (7.84) and (7.85) into (7.83) gives the function

$$v_k^\infty(t,x,y,p) = \frac{|p|^{1/2} \exp\{i(|x||p| - t\omega_k(|p|) - \frac{\pi}{4} - \frac{\pi}{4}\operatorname{sgn} U_k'(|p|))\}}{t\{U_k(|p|)\ |U_k'(|p|)|\}^{1/2}} w_k(y,p)\hat{h}_k(p) \tag{7.86}$$

To find the asymptotic wave function for $v_k(t,x,y)$ it is necessary to solve (7.79) for p and substitute in (7.86). The result may be described by means of the two inverse functions of $U_k(|p|)$ which may be defined as follows: see Figure 10.

$$\left.\begin{aligned} |p| &= P_k^f(q) \Leftrightarrow U_k(|p|) = q \text{ and } p_k \leq |p| \leq p_k^A \\ |p| &= P_k^s(q) \Leftrightarrow U_k(|p|) = q \text{ and } |p| \geq p_k^A \end{aligned}\right\} \tag{7.87}$$

It is clear from the discussion of $U_k(|p|)$ that P_k^f and P_k^s are analytic functions, P_k^f maps $\{q:\ U_k^A \leq q \leq c_2\}$ onto $\{|p|:\ p_k \leq |p| \leq p_k^A\}$ and P_k^s maps $\{q:\ U_k^A \leq q \leq c_1\}$ onto $\{|p|:\ |p| \geq p_k\}$.

The asymptotic behavior of $v_k(t,x,y)$ can now be described. The point of stationary phase $|p| = P_k^f(|x|/t)$ makes a contribution

$$v_k^{\infty,f}(t,x,y) = \chi_k^f(t,x)\ v_k^\infty(t,x,y,P_k^f(|x|/t)x/|x|) \tag{7.88}$$

where

$$\chi_k^f(t,x) \text{ is the characteristic function of } \{(t,x):\ U_k^A \leq |x|/t \leq c_2\} \tag{7.89}$$

Similarly, the point of stationary phase $|p| = P_k^s(|x|/t)$ makes a contribution

$$v_k^{\infty,s}(t,x,y) = \chi_k^s(t,x)\ v_k^\infty(t,x,y,P_k^s(|x|/t)x/|x|) \tag{7.90}$$

where

$\chi_k^s(t,x)$ is the characteristic function of

$$\{(t,x):\ U_k^A \le |x|/t \le c_1\} \tag{7.91}$$

The functions $v_k^{\infty,f}$ are called the "fast waves" because they describe waves which arrive at points (x,y) at times $t = |x|/c_2$ corresponding to the speed $c(y) = c_2$ of waves in the "fast" medium filling $y > h$. Similarly, the functions $v_k^{\infty,s}$ are called the "slow waves" because they describe waves which arrive at (x,y) at times $t = |x|/c_1$ corresponding to the speed $c(y) = c_1$ of waves in the "slow" medium filling $0 < y < h$. Finally, the total asymptotic wave function is the sum

$$v_k^\infty(t,x,y) = v_k^{\infty,f}(t,x,y) + v_k^{\infty,s}(t,x,y) \tag{7.92}$$

The following convergence theorem was proved in [40] by the method outlined in section 5.

7.15 Theorem

Let $h \in \mathcal{H}$. Then for each $k \ge 1$, $v_k^\infty(t,\cdot,\cdot) \in \mathcal{H}$ for all $t > 0$ and $t \to v_k^\infty(t,\cdot,\cdot) \in \mathcal{H}$ is continuous. Moreover, $v_k^\infty(t,\cdot,\cdot)$ is an asymptotic wave function for the modal wave $v_k(t,\cdot,\cdot) = \exp(-itA^{1/2})\, P_k h$, i.e.,

$$\lim_{t\to\infty} \| v_k(t,\cdot,\cdot) - v_k^\infty(t,\cdot,\cdot)\|_{\mathcal{H}} = 0 \tag{7.93}$$

The same methods were used in [40] to prove convergence in the energy norm when h has finite energy.

Note that $v_k^\infty(t,x,y)$ represents a guided wave which propagates radially outward in horizontal planes y = const. and is exponentially damped in the vertical coordinate y. This is evident from the defining equations (7.86), (7.88), (7.90) and (7.92).

7.16 Asymptotic wave functions for the free mode

It will now be shown that the free mode wave function $v_0(t,x,y)$ is asymptotically equal in $\mathcal{H}$ to a free wave propagating with speed c_2 in the half-space $y \ge h$. To this end note that

$$v_0(t,x,y) = \frac{1}{2\pi} \int_{\Omega_0} \exp\{i(x\cdot p - t\lambda^{1/2})\}\, w_0(y,p,\lambda)\hat{h}_0(p,\lambda)\,dp\,d\lambda \tag{7.94}$$

where $\hat{h}_0 \in L_2(\Omega_0)$. The representation (7.27) for $w_0(y,p,\lambda)$ implies that

$$v_0(t,x,y) = v_0^+(t,x,y-h) + v_0^-(t,x,y-h), \; y > h \tag{7.95}$$

where

$$v_0^+(t,x,y) = \frac{1}{2\pi} \int_{\Omega_0} e^{i(x\cdot p+y\xi-t\lambda^{1/2})} a(p,\lambda)\, \gamma_+(\xi,\eta)\hat{h}_0(p,\lambda)dpd\lambda \tag{7.96}$$

and

$$v_0^-(t,x,y) = \frac{1}{2\pi} \int_{\Omega_0} e^{i(x\cdot p-y\xi-t\lambda^{1/2})} a(p,\lambda)\, \gamma_-(\xi,\eta)\hat{h}_0(p,\lambda)dpd\lambda \tag{7.97}$$

The change of variables

$$(p,\lambda) \to (p,q), \; q = \xi = (\lambda/c_2^2 - |p|^2)^{1/2} \tag{7.98}$$

in (7.96) gives

$$v_0^+(t,x,y) = \frac{1}{(2\pi)^{3/2}} \int_{q\geq 0} e^{i(x\cdot p+yq-t\omega(p,q))}\hat{h}(p,q)dpdq \tag{7.99}$$

where

$$\hat{h}(p,q) = c_2^2\rho_2^{1/2}(2|q|)^{1/2}(\gamma_+(\xi,\eta)/|\gamma_+(\xi,\eta)|)\hat{h}_0(p,\lambda) \tag{7.100}$$

and

$$\lambda = \lambda(p,q) = \omega(p,q)^2 = c_2^2(|p|^2 + q^2) \tag{7.101}$$

Similarly, the change of variables

$$(p,\lambda) \to (p,q), \; q = -\xi = -(\lambda/c_2^2 - |p|^2)^{1/2} \tag{7.102}$$

in (7.97) gives

$$v_0^-(t,x,y) = \frac{1}{(2\pi)^{3/2}} \int_{q\leq 0} e^{i(x\cdot p+yq-t\omega(p,q))}\hat{h}(p,q)dpdq \tag{7.103}$$

where $\hat{h}$ is defined by (7.100). Adding (7.99) and (7.103) and using (7.95) shows that

$$v_0(t,x,y+h) = \frac{1}{(2\pi)^{3/2}} \int_{R^3} e^{i(x\cdot p+yq-t\omega(p,q))}\hat{h}(p,q)\,dpdq \tag{7.104}$$

for all $y \geq 0$. Moreover, (7.100) implies that $\hat{h} \in L_2(R^3)$. Thus (7.104) and (7.101) imply that in the half-space $y > h$ $v_0(t,x,y)$ coincides with a solution in $L_2(R^3)$ of the d'Alembert equation with propagation speed c_2. Now, the results of section 3 imply that the right-hand side of (7.104) has an asymptotic wave function in $L_2(R^3)$; say

$$w^\infty(t,x,y) = G(r-c_2t,\theta)/r, \quad r^2 = |x|^2 + y^2, \quad \theta = (x,y)/r \tag{7.105}$$

It follows that if

$$v_0^\infty(t,x,y) = \begin{cases} w^\infty(t,x,y-h), & y \geq h \\ 0, & 0 \leq y < h \end{cases} \tag{7.106}$$

then

$$\lim_{t\to\infty} \| v_0(t,\cdot,\cdot) - v_0^\infty(t,\cdot,\cdot)\|_{\mathcal{H}} = 0 \tag{7.107}$$

A proof may be found in [40]. This paper also contains a proof of convergence in the energy norm, when h has finite energy, and applications of these results to the calculation of asymptotic energy distributions in stratified fluids.

7.17 Other cases

The case of the symmetric Epstein profile, defined by

$$c^{-2}(y) = c_0^{-2}\,\mathrm{sech}^2(y/H) + c_\infty^{-2}\tanh^2(y/H) \tag{7.108}$$

and $\rho(y) \equiv 1$ was studied by the author in [41] where eigenfunction expansions and asymptotic wave functions are derived. Eigenfunction expansions for the case of the general Epstein profile

$$c^{-2}(y) = K\cosh^2(y/H) + L\tanh(y/H) + M \tag{7.109}$$

and $\rho(y) \equiv 1$ have been given by J. C. Guillot and the author [14, 15]. Asymptotic wave functions for this case are currently being studied by Y. Dermenjian, J. C. Guillot and the author. Preliminary studies show that the results given above for the Pekeris model are valid for a large class of profiles $c(y)$, $\rho(y)$. The essential hypotheses, apart from the boundedness (7.7), are that $c(y)$ should have a global minimum at some finite point and that $c(y)$ should tend to a limit at infinity sufficiently rapidly. If $c(y)$ does not have a minimum then there are no guided waves. However, these results have not yet been proved in this generality.

8. PROPAGATION IN CRYSTALS

Acoustic wave propagation in an unlimited homogeneous crystal is analyzed in this section. The analysis is similar to that for homogeneous fluids given in section 3. The principal new feature is the influence of anisotropy on the structure of the asymptotic wave functions.

A homogeneous crystal is characterized by a constant density $\rho(x) = \rho$ and stress-strain tensor $c^{jk}_{\ell m}(x) = c^{jk}_{\ell m}$. It will suffice to consider the case $\rho = 1$. Thus the propagation problem reduces in this case to the Cauchy problem for the system

$$\frac{\partial^2 u_j}{\partial t^2} - c^{\ell m}_{jk} \frac{\partial^2 u_\ell}{\partial x_k \partial x_m} = 0, \; j = 1,2,3 \tag{8.1}$$

where the constants $c^{\ell m}_{jk}$ satisfy (2.13) and (2.35).

8.1 Hilbert space formulation

It was shown in section 2 that the differential operator A defined by

$$(Au)_j = - c^{\ell m}_{jk} \frac{\partial^2 u_\ell}{\partial x_k \partial x_m}, \; j = 1,2,3 \tag{8.2}$$

is formally selfadjoint in the Hilbert space $\mathcal{H} = L_2(R^3, C^3)$ with inner product

$$(u,v) = \int_{R^3} \overline{u_j(x)} \, v_j(x) \, dx \tag{8.3}$$

In fact, the operator $\mathcal{A}$ in $\mathcal{H}$ with domain $D(\mathcal{A}) = \mathcal{D}(R^3)$ is essentially selfadjoint and its unique selfadjoint exten[illegible]n is the operator A defined by

$$D(A) = \mathcal{H} \cap \{u: \ \mathcal{A}u \in \mathcal{H}\} \tag{8.4}$$

$$Au = \mathcal{A}u \text{ for all } u \in D(A) \tag{8.5}$$

It is easy to verify, using the Plancherel theory of the Fourier transform, the following

8.2 Theorem

A is a selfadjoint, real positive operator in $\mathcal{H}$.

It follows, as in preceding sections, that the Cauchy problem for (8.1) has a solution in $\mathcal{H}$ of the form

$$u_j(t,x) = \text{Re } \{v_j(t,x)\} \tag{8.6}$$

where

$$v(t,\cdot) = \exp\,(-itA^{1/2})h, \quad h = f + iA^{-1/2}g \in \mathcal{H} \tag{8.7}$$

whenever the Cauchy data $u(0,x) = f(x)$ and $\partial u(0,x)/\partial t = g(x)$ satisfy $f \in \mathcal{H}$, $g \in D(A^{-1/2})$.

8.3 Fourier analysis of A

The Plancherel theory of the Fourier transform $\Phi_0 : L_2(R^3) \to L_2(R^3)$ was defined and used in section 3; see (3.12). It may be extended immediately to $\mathcal{H} = L_2(R^3, C^3)$ by defining

$$\Phi_0 u = \Phi_0(u_1,u_2,u_3) = (\Phi_0 u_1, \Phi_0 u_2, \Phi_0 u_3) \tag{8.8}$$

and Φ_0 is also unitary in $\mathcal{H}$. Property (3.14) implies that the operator $\Phi_0 A\, \Phi_0^*$ corresponds to multiplication by the 3×3 matrix valued function

$$A(p) = (A_{j\ell}(p)) = (c_{jk}^{\ell m}\, p_k p_m), \quad p \in R^3 \tag{8.9}$$

Thus

$$A = \Phi_0^* A(\cdot)\, \Phi_0 \tag{8.10}$$

Moreover, conditions (2.13) and (2.35) imply that A(p) is a real Hermitian positive definite matrix for all $p \in R^3 - \{0\}$. The spectral analysis of A will be based on (8.10). The analysis begins with

8.4 Spectral analysis of A(p)

The eigenvalues of A(p) are the roots μ of the characteristic polynomial

$$\det(\mu 1 - A(p)) = 0 \tag{8.11}$$

The Hermitian positive definiteness of A(p) implies that the roots are real and positive for all $p \in R^3 - \{0\}$. They may be uniquely defined as functions of p by enumerating them according to their magnitudes:

$$0 \le \mu_1(p) \le \mu_2(p) \le \mu_3(p) \text{ for all } p \in R^3 \tag{8.12}$$

A result of T. Kato [18] implies (see also [37])

$$\mu_j\colon\ R^3 \to R \text{ is continuous, } j = 1,2,3 \tag{8.13}$$

Equation (8.11) implies that $\mu_j(p)$ is homogeneous of degree 2

$$\mu_j(\alpha p) = \alpha^2 \mu_j(p) \text{ for all } \alpha \in R \text{ and } p \in R^3 \tag{8.14}$$

The functions

$$\lambda_j(p) = \sqrt{\mu_j(p)},\ p \in R^3,\ j = 1,2,3 \tag{8.15}$$

are also needed below. A detailed study of these functions has been made by the author in connection with a formulation of elasticity theory in terms of first order symmetric hyperbolic systems; see [29,35,37,44]. A number of results from these papers are quoted and used below.

It was shown in [44] that there exists a homogeneous polynomial $\mathcal{O}(p) \not\equiv 0$ such that the points $p \in R^3$ where two or more roots $\mu_j(p)$ coincide are contained in the cone

$$Z = \{p \in R^3 : \mathcal{O}(p) = 0\} \tag{8.16}$$

Thus

$$0 < \mu_1(p) < \mu_2(p) < \mu_3(p) \text{ for all } p \in R^3 - Z \tag{8.17}$$

It follows that

$$\mu_j(p) \text{ is analytic on } R^3 - Z,\ j = 1,2,3 \tag{8.18}$$

The orthogonal projection of C^3 onto the eigenspace for $\mu_j(p)$ is given by [18]

$$\hat{P}_j(p) = -\frac{1}{2\pi i} \int_{\gamma_j(p)} (A(p) - z1)^{-1} dz,\ j = 1,2,3 \tag{8.19}$$

where

$$\gamma_j(p) = \{z : |z - \mu_j(p)| = c_j(p)\},\ j = 1,2,3 \tag{8.20}$$

and the radii $c_j(p)$ are chosen so small that the 3 circles $\gamma_j(p)$ are disjoint. This is possible for all $p \in R^3 - Z$ by (8.17). The matrix valued functions $\hat{P}_j$ so defined can be shown to have the following properties [18,44]:

$$\hat{P}_j(p) \text{ is analytic on } R^3 - Z,\ j = 1,2,3 \tag{8.21}$$

$$\hat{P}_j(\alpha p) = \hat{P}_j(p) \text{ for all } \alpha \neq 0 \tag{8.22}$$

$$\hat{P}_j(p)^* = \hat{P}_j(p),\ \hat{P}_j(p)\, \hat{P}_k(p) = \delta_{jk}\hat{P}_k(p) \text{ for } p \in R^3 - Z \tag{8.23}$$

$$\sum_{j=1}^{3} \hat{P}_j(p) = 1 \text{ for } p \in R^3 - Z \tag{8.24}$$

$$A(p)\, \hat{P}_j(p) = \mu_j(p)\, \hat{P}_j(p) \text{ for } p \in R^3 - Z,\ j = 1,2,3 \tag{8.25}$$

The last two properties imply that the projections $\hat{P}_j(p)$ define a spectral representation for $A(p)$; i.e.,

$$A(p) = \sum_{j=1}^{3} \mu_j(p)\, \hat{P}_j(p) \quad \text{for } p \in R^3 - Z \tag{8.26}$$

8.5 Spectral analysis of A

The representations (8.10) and (8.26) provide a complete spectral analysis of A. In particular, it follows that A is an absolutely continuous operator whose spectrum is $[0,\infty)$ (cf. [36, 44]). Moreover, if $\Psi(\mu)$ is any bounded Lebesgue-measurable function of $\mu \geq 0$ then

$$\Psi(A) = \Phi_0^* \sum_{k=1}^{3} \Psi(\mu_k(\cdot))\, \hat{P}_k(\cdot)\, \Phi_0 \tag{8.27}$$

8.6 Solution in $\mathcal{H}$ of the Cauchy problem

Application of (8.27) to the solution in $\mathcal{H}$ (8.7) yields the representation

$$v(t,x) = \sum_{k=1}^{3} v_k(t,x) \tag{8.28}$$

where

$$v_k(t,x) = \frac{1}{(2\pi)^{3/2}} \int_{R^3} e^{i(x\cdot p - t\lambda_k(p))}\, \hat{P}_k(p)\hat{h}(p)\,dp \tag{8.29}$$

and $\lambda_k(p) = \sqrt{\mu_k(p)}$. Of course, the integral in (8.29) converges in $\mathcal{H}$, in the sense of the Plancherel theory, rather than pointwise. Equations (8.28), (8.29) represent solutions in $\mathcal{H}$ of (8.1) as a superposition of solutions

$$e^{i(x\cdot p - t\lambda_k(p))}\, \hat{P}_k(p)\hat{h}(p) \tag{8.30}$$

This may be interpreted as a plane wave which propagates in the crystal with direction $p/|p|$, wave number $|p|$ and frequency

$$\omega = \lambda_k(p) \tag{8.31}$$

The polarization of the wave is determined by $\hat{P}_k(p)$. The corresponding generalized eigenfunctions of A are the matrix plane waves [30]

$$w_k(x,p) = \frac{1}{(2\pi)^{3/2}} \exp(ix\cdot p)\, \hat{P}_k(p) \tag{8.32}$$

8.7 The dispersion relation, phase and group velocities

The dispersion relation between the frequency ω and wave vector p of plane waves in the crystal is (8.31) or, by (8.15) and (8.11)

$$\det(\omega^2 1 - A(p)) = 0 \tag{8.33}$$

The phase velocity for (8.1) is

$$v_{ph}(p) = \frac{\omega}{|p|} \cdot \frac{p}{|p|} = \frac{\lambda_k(p)}{|p|^2}\, p = \lambda_k\left(\frac{p}{|p|}\right) \frac{p}{|p|} \tag{8.34}$$

by the homogeneity of $\lambda_k(p)$. The group velocity for (8.1) is

$$v_g(p) = \nabla_p \omega = \nabla_p \lambda_k(p) \tag{8.35}$$

The medium is said to be isotropic if $v_{ph}(p)$ and $v_g(p)$ have the same direction for all $p \in R^3 - \{0\}$. Otherwise it is said to be anisotropic. It is easy to verify that the medium is isotropic if and only if $\lambda_k(p)$ is a function of $|p|$ alone. In this case $\lambda_k(p) = c_k|p|$ and $v_{ph}(p) = v_g(p) = c_k p/|p|$.

The phase and group speeds for (8.1) are the magnitudes of the corresponding velocities. Thus

$$\left.\begin{aligned} c_{ph}(p) &= |v_{ph}(p)| = \lambda_k(p/|p|) \\ c_g(p) &= |v_g(p)| = \nabla_p \lambda_k(p) \end{aligned}\right\} \tag{8.36}$$

Note that both are homogeneous of degree zero in p and hence depend only on the direction of propagation $p/|p|$. The anisotropy of the medium characterized by (8.1) can be visualized by means of

8.8 The slowness surface S

This is the real algebraic variety defined by

$$S = \{p \in R^3: \det (1 - A(p)) = 0\} \tag{8.37}$$

It is clear from the definition of the $\lambda_k(p)$ that

$$S = \bigcup_{k=1}^{3} S_k \tag{8.38}$$

where

$$S_k = \{p \in R^3: \lambda_k(p) = 1\} \tag{8.39}$$

or, by (8.36) and the homogeneity of $\lambda_k(p)$,

$$S_k = \{p \in R^3: |p|c_k(p) = 1\} \tag{8.40}$$

Thus $p \in S$ if and only if $|p|$ is the reciprocal of a phase speed for the direction p. Note that the slowness surface of an isotropic medium is a set of concentric spheres with centers at the origin.

The properties of slowness surfaces were studied in [35] and [44]. In particular, the following properties were established

$$S_k \text{ is continuous and star-shaped with respect to } 0 \tag{8.41}$$

As an algebraic variety, S will in general have singular points and these are precisely the set

$$Z'_S = \{p \in S: p \in S_j \cap S_k \text{ for some } j \neq k\} \tag{8.42}$$

Hence

$$S_k - Z'_S,\ k = 1,2,3, \text{ are disjoint and analytic} \tag{8.43}$$

8.9 The wave surface W

The variation of the phase speed with direction is represented by the slowness surface S. Similarly, the variation of the group speed is represented by the wave surface W. W may be defined as the polar reciprocal of S with respect to the unit sphere. This means that

$$W = \{x \in R^3: \; x \cdot p = 1 \text{ is a tangent plane to } S\} \tag{8.44}$$

It is known that W is a real algebraic variety whose degree is the class number of S [7,28]. Moreover, the relation of S and W is symmetric: S is also the polar reciprocal of W. It is clear that if

$$N(p) = \text{the } \underline{\text{set}} \text{ of all exterior unit normals to } S \text{ at } p \tag{8.45}$$

then

$$W = \{x = (p \cdot N(p))^{-1} N(p): \; p \in S\} \tag{8.46}$$

Now the group velocity $v_g(p) = \nabla_p \lambda_k(p)$ is normal to S at each $p \in S - Z_S'$. Moreover, $p \cdot \nabla_p \lambda_k(p) = \lambda_k(p) = 1$ for such points p by (8.39) and the homogeneity of $\lambda_k(p)$. Hence

$$\{x = v_g(p) \equiv \nabla_p \lambda_k(p): \; p \in S - Z_S'\} \subset W \tag{8.47}$$

for k = 1,2,3.

8.10 The polar reciprocal map T: $S \to W$

This is the map defined in (8.46); i.e.,

$$T(p) = (p \cdot N(p))^{-1} N(p) \text{ for all } p \in S \tag{8.48}$$

As indicated above, N(p) is not, in general, single valued. It follows that T may be neither single-valued nor injective. However, it was shown in [49] that if

$$\left.\begin{aligned} Z_S' &= \text{set of singular points of } S \\ Z_W' &= \text{set of singular points of } W \end{aligned}\right\} \tag{8.49}$$

$$Z_S'' = T^{-1} Z_W', \; Z_W'' = T Z_S' \tag{8.50}$$

$$Z_S = Z_S' \cup Z_S'', \; Z_W = Z_W' \cup Z_W'' \tag{8.51}$$

then Z_S and Z_W are sub-varieties of dimension ≤ 1 and

$$T \text{ is bijective and analytic from } S - Z_S \text{ to } W - Z_W \tag{8.52}$$

8.11 Examples

The equation (8.37) for the *slowness surface of a crystal* contains 21 independent parameters in the most general case (triclinic crystals). Hence a great variety of slowness surfaces are possible. Crystal symmetries may reduce the number of parameters. The slowness surfaces of the various symmetry classes have been studied by many authors. Thorough discussions and examples may be found in [3] and [24] where specific numerical information on the stress-strain tensors of real crystals may also be found. Here two examples will be described briefly to show the kind of surfaces that may occur.

Cubic crystals. In this case symmetry reduces the number of independent parameters to 3 and the equation for S can be written [24]

$$\sum_{j=1}^{3} \frac{p_j^2}{a-b|p|^2-cp_j^2} = 1 \tag{8.53}$$

Of course, the positive definiteness of $c_{jk}^{\ell m}$ imposes certain numerical restrictions on a, b and c. Equation (8.53) represents a surface of degree 6 which is irreducible except for special parameter values.

Hexagonal crystals. In this case symmetry reduces the number of independent parameters to 5. Moreover, S is necessarily a surface of revolution and reduces to two components whose equations can be written [24]

$$a^2(p_1^2 + p_2^2) + b^2p_3^2 = 1 \tag{8.54}$$

$$\frac{p_1^2+p_2^2}{c^2-d^2|p|^2+e(p_1^2+p_2^2)} + \frac{p_3^2}{c^2-d^2|p|^2+fp_3^2} = 1 \tag{8.55}$$

(where a, b, c, d, e and f can be expressed in terms of 5 independent parameters). The two equations have degrees 2 and 4, respectively. These surfaces of revolution can be visualized from their traces on the p_1,p_3-plane; see [24,p.99] for a graph of such an S and the corresponding W. It is seen that in the example Z_S' consists of 2 circles and 2 points lying in S while Z_W' consists of 8 circles and 2 points lying in W.

8.12 Asymptotic wave functions for crystals

It was shown in [44] that the equations (8.1) for acoustic waves in crystals have asymptotic wave functions of the form

$$v^{\infty}(t,x) = \sum_{\alpha=1}^{\nu(\theta)} F(x\cdot s^{(\alpha)}(\theta) - t, s^{(\alpha)}(\theta))/|x|, \qquad x = |x|\theta \tag{8.56}$$

where

$$s^{(\alpha)}(\theta) \in S, \ \alpha = 1,2,\cdots,\nu(\theta) \tag{8.57}$$

is the solution set of the equation

$$N(s) = \theta \tag{8.58}$$

Thus $s^{(\alpha)}$ defines the multivalued inverse of the Gauss map N of S. The principal properties of $v^{\infty}(t,x)$ are described by the following theorem whose proof is contained in [44].

8.13 Theorem

For each $h \in \mathcal{H}$ there exists a unique $F: \ R \times S \to C^3$ such that

$$v^{\infty}(t,\cdot) \in \mathcal{H} \text{ for all } t \in R \tag{8.59}$$

$$t \to v^{\infty}(t,\cdot) \in \mathcal{H} \text{ is continuous for all } t \in R \tag{8.60}$$

$$\|v^{\infty}(t,\cdot)\|_{\mathcal{H}} \leq C\|h\|_{\mathcal{H}} \text{ where C is independent of h and t} \tag{8.61}$$

Finally, v^{∞} is an asymptotic wave function for $v(t,\cdot) = \exp(-itA^{1/2})h$:

$$\lim_{t\to\infty} \|v(t,\cdot) - v^{\infty}(t,\cdot)\|_{\mathcal{H}} = 0 \tag{8.62}$$

Moreover, explicit constructions of $s^{(\alpha)}(\theta)$ and $F(\tau,s)$ are given in [49]. In the present case they take the following form.

8.14 Construction of $s^{(\alpha)}(\theta)$

The construction consists of two steps.

$$x^{(\alpha)}(\theta),\ \alpha = 1,\cdots,\nu(\theta) \text{ is the intersection of } W - Z_W \text{ and the ray from } 0 \text{ along } \theta \tag{8.63}$$

$$s^{(\alpha)}(\theta) = T^{-1}x^{(\alpha)}(\theta) \in S - Z_S \tag{8.64}$$

Note that this defines $s^{(\alpha)}(\theta)$ for all θ outside of the null set

$$Z^0 = \{\theta:\ x = |x|\theta \in Z_W\} \subset S^2 = \{\theta:\ |\theta| = 1\} \tag{8.65}$$

8.15 Construction of $F(\tau,s)$

F is calculated from $h = v(0,\cdot) \in \mathcal{H}$ by the rule

$$F(\tau,s) = (2\pi)^{-1/2}\ \Psi(s) \int_0^\infty e^{i\tau\lambda}\ \hat{h}(\lambda s)\lambda d\lambda \tag{8.66}$$

where

$$\Psi(s) = \psi(s)|K(s)|^{-1/2}\ |T(s)|^{-1}\ \hat{P}(s) \tag{8.67}$$

$$\psi(s) = \exp\{i\ \frac{\pi}{4}\ (p^{-}(s) - p^{+}(s))\} \tag{8.68}$$

$$p^{\pm}(s) = \text{the number of principal curvatures of } S \text{ at } s \text{ which are } \gtrless 0. \tag{8.69}$$

$$K(s) = \text{Gaussian curvature of } S \text{ at } s \tag{8.70}$$

$$\hat{P}(s) = \text{orthogonal projection of } C^3 \text{ onto the eigenspace for the eigenvalue } \mu = 1 \text{ of } A(s)\ (s \in S) \tag{8.71}$$

It is shown in [44] that $\Psi(s)$ is defined for all $s \in S - Z_S$. In particular, the parabolic points of S lie in Z_S. The integral for F need not converge pointwise, but it converges in the Hilbert space $\mathcal{H}(S)$ with norm defined by

$$\|F\|^2_{\mathcal{H}(S)} = \int_0^\infty \int_S |F(\tau,s)|^2 \, |K(s)T(s)| \, dS d\tau \qquad (8.72)$$

Moreover, the operator Θ: $\mathcal{H} \to \mathcal{H}(S)$ defined by $\Theta h = F$ is an isometry.

8.16 Propagation in non-uniform crystals

The method developed in [42] and section 4 can be applied to local perturbations of uniform crystals. Eigenfunction expansions for non-uniform crystals, and more general systems, have been given by G. Nenciu [25].

REFERENCES

1. S. Agmon, Lectures on Elliptic Boundary Value Problems, New York: Van Nostrand, 1965.
2. O. Arena and W. Littman, 'Farfield' behavior of solutions to partial differential equations, Ann. Scuola Norm. Sup. Pisa 26, 807-827 (1972).
3. B. A. Auld, Acoustic Fields and Waves in Solids, Vols. I and II, New York: J. Wiley and Sons, 1973.
4. L. Brillouin, Les Tenseurs en Mécanique et en Elasticité, New York: Dover, 1946.
5. L. Brillouin, Wave Propagation and Group Velocity, New York: Academic Press, 1960.
6. R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. 2, New York: Interscience Publishers, 1962.
7. G. F. D. Duff, The Cauchy problem for elastic waves in an anisotropic medium, Phil. Trans. Roy. Soc. London, Series A 252, 249-273 (1960).
8. F. G. Friedlander, Sound Pulses, Cambridge: Cambridge University Press, 1958.
9. C. Goldstein, Eigenfunction expansions associated with the Laplacian for certain domains with infinite boundaries, Trans. Amer. Math. Soc. 135, 1-50 (1969).
10. C. Goldstein, The singularities of the S-matrix and Green's function associated with perturbations of $-\Delta$ acting in a cylinder, Bull. Amer. Math. Soc. 79, 1303-1307 (1973).
11. C. Goldstein, Meromorphic continuation of the S-matrix for the operator $-\Delta$ acting in a cylinder, Proc. Amer. Math. Soc. 42, 555-562 (1974).
12. C. Goldstein, Analytic perturbations of the operator $-\Delta$, J. Math. Ann. Appl. 25, 128-148 (1969).
13. J. C. Guillot and C. H. Wilcox, Théorie spectrale du laplacian dans des ouverts coniques et cylindriques non bornés, C. R. Acad. Sci. Paris 282, 1171-1174 (1976).

14. J. C. Guillot and C. H. Wilcox, Théorie spectrale de l'opérateur d'Epstein (théorie des guides d'onde ouverts), C. R. Acad. Sci. Paris 281, 399-402 (1975).
15. J. C. Guillot and C. H. Wilcox, Spectral analysis of the Epstein operator, ONR Technical Summary Rept. #27, University of Utah, August 1975.
16. T. Ikebe, Eigenfunction expansions associated with the Schrödinger operator and their application to scattering theory, Arch. Rational Mech. Anal. 5, 1-34 (1960).
17. D. S. Jones, The eigenvalues of $\nabla^2 u + \lambda u = 0$ when the boundary conditions are given on semi-infinite domains, Proc. Cambridge Phil. Soc. 49, 668-684 (1953).
18. T. Kato, Perturbation Theory for Linear Operators, New York: Springer, 1966.
19. J. L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, New York: Springer, 1972.
20. A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity, 4th Edition, New York: Dover, 1944.
21. W. C. Lyford, Spectral analysis of the Laplacian in domains with cylinders, Math. Ann. 218, 229-251 (1975).
22. W. C. Lyford, Asymptotic energy propagation and scattering of waves in waveguides with cylinders, Math. Ann. 219, 193-212 (1976).
23. M. Matsumura, Asymptotic behavior at infinity for Green's functions of first order systems with characteristics of nonuniform multiplicity, Publ. R.I.M.S., Kyoto Univ., Ser. A (to appear).
24. M. J. P. Musgrave, Crystal Acoustics, San Francisco: Holden-Day, 1970.
25. G. Nenciu, Eigenfunction expansions for wave propagation problems in classical physics, preprint, 1975.
26. C. L. Pekeris, Theory of propagation of explosive sound in shallow water, Geol. Soc. Am., Memoir 27 (1948).
27. F. Rellich, Das Eigenwertproblem von $\Delta u + \lambda u = 0$ in Halbröhren, in Studies and Essays Presented to R. Courant, New York: Interscience, 1948.
28. G. Salmon, Analytic Geometry of Three Dimensions, Vol. 2, Fifth Edition, Dublin, 1914. Reprinted New York: Chelsea Publishing Co., 1965.
29. J. R. Schulenberger and C. H. Wilcox, A coerciveness inequality for a class of nonelliptic operators of constant deficit, Ann. Mat. Pura Appl. 92, 77-84 (1972).
30. J. R. Schulenberger and C. H. Wilcox, Eigenfunction expansions and scattering theory for wave propagation problems of classical physics, Arch. Rational Mech. Anal. 46, 280-320 (1972).
31. E. Skudrzyk, The Foundations of Acoustics, New York: Springer, 1971.
32. M. Vishik and O. A. Ladyzhenskaya, Boundary value problems for partial differential equations and certain classes of operator equations, Uspehi Mat. Nauk. (N.S.) 11, 41-97 (1956) [Russian]. A.M.S. Translations, Series II, 10, 223-281 (1958).

33. C. H. Wilcox, Initial-boundary value problems for linear hyperbolic partial differential equations of the second order, Arch. Rational Mech. Anal. 10, 361-400 (1962).
34. C. H. Wilcox, The domain of dependence inequality for symmetric hyperbolic systems, Bull. Amer. Math. Soc. 70, 149-154 (1964).
35. C. H. Wilcox, Wave operators and asymptotic solutions of wave propagation problems of classical physics, Arch. Rational Mech. Anal. 22, 37-78 (1966).
36. C. H. Wilcox, Transient wave propagation in homogeneous anisotropic media, Arch. Rational Mech. Anal. 37, 323-343 (1970).
37. C. H. Wilcox, Measurable eigenvectors for Hermitian matrix-valued polynomials, J. Math. Anal. Appl. 40, 12-19 (1972).
38. C. H. Wilcox, Scattering states and wave operators in the abstract theory of scattering, J. Functional Anal. 12, 257-274 (1973).
39. C. H. Wilcox, Spectral analysis of the Laplacian with a discontinuous coefficient, Proc. of C.I.M.E. Conf. on Spectral Analysis - Sept. 1973, 233-253, Rome: Edizioni Cremonesa, 1974.
40. C. H. Wilcox, Transient electromagnetic wave propagation in a dielectric waveguide, Proc. of Conf. on the Mathematical Theory of Electromagnetism, Istituto Nazionale di Alta Matematica, Rome, 1974.
41. C. H. Wilcox, Transient acoustic wave propagation in a symmetric Epstein duct, ONR Technical Summary Rept. #25, University of Utah, May 1974.
42. C. H. Wilcox, Scattering Theory for the d'Alembert Equation in Exterior Domains, Lecture Notes in Mathematics, Vol. 442, New York: Springer, 1975.
43. C. H. Wilcox, Spectral analysis of the Pekeris operator in the theory of acoustic wave propagation in shallow water, Arch. Rational Mech. Anal. 60, 259-300 (1976).
44. C. H. Wilcox, Asymptotic wave functions and energy distributions in strongly propagative anisotropic media, ONR Technical Rept. #28, University of Utah, March 1976.